Electing a Proportional Committee with Majority Judgment ballots

Mémoire d'Initiation à la Recherche

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Abstract

The Majority Judgment is a voting system in which each voter is asked to assign a grade to every candidate independently. It also states how the grades received by each candidate should be aggregated in order to grant them a final grade. This system permits to rank all candidates with respect to their final grades and to determine a winner. The Majority Judgment is hence a single-winner voting system. The aim of the present article is to extend this kind of voting to a proportional multi-winner election. We want to elect a fixed-size set of winners, called a parliament or a committee, on the basis of grade ballot papers while respecting proportionality. First, we specify what is a voting rule in such a context and which general axioms it should verify. Then, we extend the concept of proportionality to the setting of grade ballots, giving birth to a fundamental axiom (Epp) and even some voting rules (VGM, INP, LUD). This analysis also makes us realize that the traditional proportional allocation does not necessarily lead to the desired concept of proportionality. In order to enlarge this concept, we finally adopt a welfarist approach, through which proportionality axioms (Eup, Plp, EPlp) and other election rules (RC, TGV) are designed.

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Introduction

A voting system aims to elect one or several representatives based on voters' preferences. Every voting system proceeds in two steps: we should make a clear distinction between the mechanism by which we represent those preferences, through the form of ballot papers, and the way we aggregate them to determine a collective preference or a social decision. The first step could be called the voting form, it determines how voters can express their respective preferences. Some voting forms are ordinal, voters are asked to rank several political options, that are candidates or parties, and the others are cardinal, voters are asked to give their opinion on each option separately [Baujard and al, 2017]. The second step is usually called the election rule, it states how many candidates should be elected, and it states how to transform the ballot papers into a political representation: either a unique winner for single-winner election rules or a winning parliament for multi-winner ones. Historically, mathematicians, economists and researchers in social choice theory focused on single-winner rules based on ordinal voting, also called positional voting. Among them, the most famous are [de Borda, 1781] who invented the Borda's count, [de Condorcet, 1785] for his paradox of intransitive collective preferences based on transitive individual ones, [Arrow, 1951] for his impossibility theorem regarding social welfare functions, and [Gibbard, 1973] and [Satterthwaite, 1975] for originating another impossibility result, regarding social choice functions. Then, cardinal voting became at the heart of interest, notably with approval voting, score voting and Majority Judgment, still in a single-winner setting. Cardinal voting is interesting to the extent that it avoids the previously mentioned impossibility results [Vasiljev, 2008]. Very recently, the social choice community shifts from the single-winner to the multi-winner analysis, but only for positional [Elkind and al, 2017] and approval voting [Aziz and al, 2017] [Sánchez-Fernández and al, 2017] and not for score voting or Majority Judgment voting. This is why we propose to extend the Majority Judgment voting to the multi-winner setting.

Regarding the form that could be taken by cardinal ballot papers, we usually distinguish three types of voting: the plurality voting, the approval voting, and the score voting. In the plurality voting, each voter may vote for a unique option, and only one. It is the voting form that is the most used worldwide, by far. On the contrary, approval voting enables each voter to vote for as many candidates as she wants. For example, if there are ten candidates, a voter can approve four of them and hence disapprove the six others. With such a decision, each of the four approved candidates will receive one vote from this voter. The ballot paper which is commonly used for such a voting form is a list of all candidates (or all parties) where each option is accompanied with a tick box that can be filled (for approving) or not (for disapproving). Approval voting is currently used for municipal elections in Fargo (North Dakota) and in Saint-Louis (Missouri), and has been rigorously studied by [Brams and Fishburn, 1978]. Notably, they found that it permits to avoid the "useful vote" effect where some voters are ready to vote for a less preferred candidate for being more likely to win. Then, with score voting, preferences can be expressed even more precisely since each voter may assign a numerical score to each candidate. Generally, scores that can be granted are integers between 0 and 9. Score voting has already been used by the Republic of Venice to elect the Doge of Venice from the 13th to the 18th century. When there are three political alternatives, [Smith, 2006] showed that score voting does not encourage voters to grade a less preferred option over a more-preferred one in a strategic perspective. Finally, we can notice that approval voting

can be viewed as a particular case of score voting where there are only two scores (score of 0 for disapproved candidates and 1 for the approved ones), and further, plurality voting can be seen as a particular case of approval voting when the score of 1 can only be assigned once.

Recently, a new kind of voting has been proposed by [Balinski and Laraki, 2007] which is called the Majority Judgment (MJ). In this voting system, voters may assign a grade to each candidate among several possibilities previously defined by a common language, denoted here Ω. Grades can be numerical values, but they can take a literal form too. For instance, the common language could be composed of real numbers between 0 and 10, as well as words such as “Good”, “Insufficient” or “Bad”. Grades must necessarily be hierarchized. When they are numerical values, the hierarchy is trivial. However, when they are literal values, the common language should be arbitrarily ordered. For example, if the common language allows voters to use the grades “Green”, “Yellow” or “Red”, it should specify how these grades are ordered. The MJ voting is even more general than score voting given that score voting is included in it. As a result, the MJ voting embraces all the previously mentioned forms of cardinal voting. Majority Judgment has been already used in France by some political organizations, notably by the Primaire.org in 2017 and by the Primaire Populaire in 2022, in order to stand a candidate to the presidential election. Throughout our analysis, we will use a numerical common language, for being computable and tractable. However, all notions, axioms and methods will be applicable for literal values too, by previously defining an injective function $\varphi : \Omega \rightarrow \mathbb{R}$ which associates a real number to any grade contained in the common language. This function should respect the hierarchy structuring the common language: if $\alpha \succ \beta$, meaning that α is a higher grade than β, then we should have $\varphi(\alpha) > \varphi(\beta)$. We assume that this function could also be used for numerical values, but in this case, it should change nothing: $\varphi(\alpha) = \alpha$ for all grades $\alpha \in \Omega \subseteq \mathbb{R}$.

The Majority Judgment is not only a form of voting, but it is also an election rule: it states how to transform such ballot papers into a ranking of all candidates in order to elect a unique winner. In order to rank the candidates, they should be granted a final grade, aggregating all the grades they receive from each voter. This final grade could have been the mean grade or the minimum grade for instance. But the MJ election rule states that the final grade a candidate should be granted corresponds to her median grade. In the case where the number of voters n is even, each candidate should be assigned her lower median grade. This final grade has been called the **majority-grade**. Theoretically, the choice of the median grade implies that, for every candidate, less than 50% of voters give her a higher final grade and less than 50% of voters give her a lower final grade. Therefore, it is a stable situation, since there is no consistent majority of voters who can propose a higher or a lower final grade. Moreover, the majority-grade permits to avoid strategic behaviors when it comes to grade a candidate. Imagine that a voter assigns a grade α_1 to a candidate that is higher to her final grade $\hat{\alpha}$. A strategy for this voter could be to increase the grade she gives to that candidate in order to rise her final grade. Suppose the most strategic choice is to grant $\alpha_2 > \alpha_1$. But using the median grade for the final grade, it would change nothing, since $\hat{\alpha}$ would always be in the middle. It would only change the position of grades between α_1 and α_2. The same result can be obtained when a voter wants the final grade to be lowered. The only voter(s) who can change the majority-grade is the one(s) who gives the median grade, but they would not be better

off doing that. This is why the Majority Judgment is strategy-proof-in-grading.
After giving each candidate her majority-grade, it is possible to rank them. Nevertheless, a disadvantage of Majority Judgment is that there can easily be ties between candidate. Indeed, the median grade always corresponds to a grade contained in the common language. Thus, there are as many majority-grades as "basic" grades (those that can be assigned to a candidate). On the contrary, if the final grade had been the mean grade, there would have been an infinity of possible final grades. For instance, with $\Omega = \{3; 5; 7\}$, the mean grade can take any value between 3 and 7, whereas the median grade can only take three values: 3, 5 and 7. Yet, the less possibilities of majority-grades there are, the more likely ties will occur. Fortunately, the founders of the MJ find a solution to break all possible ties. They call it the **majority-ranking**. It states that, when two candidates receive the same majority-grade, we should remove it from their respective set of grades and take the median grade of the set that is obtained. For instance, if $\{3;\ 4;\ 5;\ 6;\ 7\}$ is the set of grades of one candidate, her (first) majority-grade is 5, so we remove it from the set to obtain $\{3;\ 4;\ 6;\ 7\}$ in which the majority-grade is 4. If both candidates always receive the same majority-grade after this process, we should iterate it until the tie is broken. If after having removed the $(n-1)$ first majority-grades, both candidates are always tied, this method does not permit to differentiate them, but such a result also means that these candidates have exactly received the same set of grades. Thus, whatever the way we choose to aggregate individual grades, both candidates would always have the same final grade and it would be impossible to rank them. Consequently, the only failure of the majority-ranking is not linked to the method in itself, but is directly implied by a specific profile of voters' expressed preferences. R. Laraki and M. Balinski also invent the **majority-value**, which is specific to each candidate. The majority-value of a candidate is the ordered sequence of her majority-grades, that is to say, her first majority-grade, following by her second one, and again until her last one. For example, if a candidate receives the set of grades $\{3;\ 4;\ 5;\ 6;\ 7\}$, her majority-value is $\{5;\ 4;\ 6;\ 3;\ 7\}$. It reorders the initial set of grades in order to facilitate the ranking between all candidates.

Our aim is to extend the Majority Judgment to the multi-winner setting, while considering proportionality. However, we should precise that our extension will only concern the ballot form implied by the Majority Judgment, and not its associated election rule. Indeed, the MJ election rule is a single-winner election rule, determining a winning option after ranking all of them. If options are candidates, this election rule could be easily extended to a multi-winner setting by giving the k seats to the k candidates who obtained the highest majority-grades. But with such an extension, proportionality is totally omitted. Even worse, when options are parties, the MJ election rule would be able to rank all of them but would say absolutely nothing on how many seats should be granted to each party and why. Thus, it seems necessary to design at least one new election rule that would be able to generate a parliament respecting proportionality statements. This is why we will only focus on MJ ballots and not on the entire MJ voting system. In other words, our goal is to find multi-winner election rule(s) that transforms MJ ballot papers, that could be named grade ballots, into a proportional parliament.

Regarding the election rules, it is widely used to model them as functions, where the inputs are ballot papers, and where the output depends on the kind of rules we consider. We usually distinguish two types of election rules: social welfare functions, which output a ranking of all possible options, and social choice functions, which output the winning

option directly. If ranking options permits to determine a winner, determining a winner does not permit to rank options. This is why social welfare functions are often preferred to social choice functions. Furthermore, the nature of the output also depends on the fact we are either in a single-winner or in a multi-winner election. When we want to elect one candidate, social choice functions return one winning candidate and social welfare functions return a ranking between all candidates. When we want to elect several candidates, social choice functions return a winning committee and social welfare functions return a ranking of all possible committees. Given that functions return very different outcomes when we switch to a multi-winner analysis, it is reasonable to give them new names. For approval voting, we already talk about Approval-Based Committee (ABC) rules [Lackner and Skowron, 2018], and thus, we distinguish ABC ranking rules from ABC choice rules. For an analysis based on grade ballots, we suggest to define Grade-Based Committee (GBC) rules, divided into GBC ranking rules and GBC choice rules, which respectively return a ranking of committees and a winning committee, on the basis of a MJ profile.

Throughout the article, we denote $V = \{v_1, \dots, v_n\}$ the set of n voters, $C = \{c_1, \dots, c_m\}$ the set of m candidates and $W \subset C$ the winning committee composed of k members. In the approval setting, for a given voter v_i, we usually distinguish approved candidates from disapproved ones, and we denote $A_i \subseteq C$ the approval subset of voter v_i with $i \in [n] = [\![1; n]\!]$, that is to say, candidates that she approves. Within the MJ approach, we will distinguish candidates with respect to their grade. Thus, we define $M_i(\alpha) \subseteq C$ the α-subset for the voter v_i, gathering the candidates who receive a grade $\alpha \in \Omega$ from this voter. We have $\bigcup_{\alpha \in \Omega} M_i(\alpha) = C$ (every candidate must receive a grade from each voter) and $M_i(\alpha) \cap M_i(\beta) = \emptyset$ (a candidate cannot receive more than one grade from each voter). We also define $M_i = (M_i(\alpha))_{\alpha \in \Omega}$ the vector of α-subsets for the voter v_i, collecting all her preferences, and $M = (M_i)_{1 \leq i \leq n}$ the MJ profile, which contains all the information about voters' expressed preferences when they face MJ ballots. Finally, let consider $W \cap M_i(\alpha)$ the subset of committee members that receive a grade α from the voter v_i. Such members are called α-representatives of voter v_i.

1 Axiomatic Approach

We denote F a GBC choice rule. We decide to focus on this kind of rule when designing some desirable axioms as it is more appropriated. In addition, all axioms that are applicable to choice rules are also for ranking rules since every ranking rule can be seen as a choice rule which returns the top-ranked option as the winner. Further, we decide to consider party-list elections, where political alternatives are parties, again for suitability. Let R the number of parties. Consequently, F is a function which returns a winning committee W for every MJ profile M, and if the r^{th} party denoted P_r is α-graded by a voter v_i, we can write $P_r \subset M_i(\alpha)$.

Further, let $s_r(W)$ the numbers of seats granted to the r^{th} party within the committee W, let $g_i(r, M)$ the grade given to that party by voter v_i in the profile M and let $h_i(r, M)$ the corresponding relative grade. We define a relative grade as follows:

$$h_i(r, M) = \frac{g_i(r, M)}{\sum_{t=1}^{R} g_i(t, M)}$$

Or, more generally, since grades are not necessarily numerical:

$$h_i(r, M) = \frac{\varphi(g_i(r, M))}{\sum_{t=1}^{R} \varphi(g_i(t, M))}$$

Finally, $h_i(r, M)$ is the "proportion of votes" granted by the voter v_i to the r^{th} party. The higher it is, the more this voter supports that party relatively to the others. Such a value will prove to be very useful when designing several following axioms.

1.1 Elementary axioms

First, a GBC choice rule must respect some basic axioms, which can be seen as natural axioms. Here, we suggest to focus on four of them: anonymity, neutrality, unanimity and monotonicity.

1.1.1 Anonymity

Each voter should be given the same consideration. Precisely, the set of grades established by a voter v_i must have the same impact than the one established by another voter v_j. It implies that if we permute the set of grades of both voters, the social choice should remain unchanged. For instance, if voter v_i and v_j respectively assigns the grades $\{\alpha, \beta, \gamma\}$ and $\{\gamma, \beta, \alpha\}$ to the set of candidates $C = \{c_1, c_2, c_3\}$ in the profile M, then the profile M' obtained from M by permuting both voters' sets of grades should lead to the same winning committee: $F(M) = F(M')$. This is anonymity. More generally, anonymity states that F must return the same output for any couple of MJ profiles (M, M') as long as they contain the same vectors M_i, that is to say, the same individual preferences, but not necessarily in the same order.

Axiom 1: Anonymity. A GBC choice rule F is anonymous if, for any permutation of voters $\pi : [n] \to [n]$ and for any profiles (M, M') such that $M'_i = M_{\pi(i)}$, we have:

$$F(M) = F(M')$$

1.1.2 Neutrality

Equivalently, each political option should be given the same consideration. Notably, it is inconceivable to give one party more seats than another if they receive the same set of grades. Further, if two political parties r and t are respectively given the grades $\{\alpha, \alpha, \alpha, \alpha, \alpha\}$ and $\{\beta, \beta, \beta, \beta, \beta\}$ by the voters $V = \{v_1, \ldots, v_5\}$ within the profile M, and if F selects a winning committee W in which both parties are respectively given s_α and s_β seats, then the profile M' obtained from M by permuting both parties' sets of grades should lead to a modified winning committee W' in which s_β seats are given to party r, s_α seats are given to the party t and where the allocation of seats between the other parties remains unchanged: $s_r(W) = s_t(W')$, $s_t(W) = s_r(W')$, $s_p(W) = s_p(W')$ for every $p \in [R] \setminus \{r, t\}$. This is neutrality. More generally, neutrality applies when two profiles display the same sets of grades assigned to the parties, but where each party not necessarily receives the same set of grades. In this case, neutrality states that for each set of grades is associated a specific number of seats, and that any party receiving a certain set of grades must be given the corresponding number of seats.

Axiom 2: Neutrality. A GBC choice rule F is neutral if, for any permutation of political parties $\sigma : [R] \to [R]$, for any profiles M and for every party r,

$$s_r(F(M')) = s_{\sigma(r)}(F(M))$$

Where M' is the permuted profile of M with respect to σ such that, for every party r:

$$P_r \subset \bigcap_{i=1}^{n} M'_i(g_i(\sigma(r), M))$$

1.1.3 Unanimity

It is quite obvious that, in the case where all voters have exactly the same preferences, that is to say, each voter gives exactly the same grade to the same party, then the winning committee should reflect exactly this common individual preference. It is the unanimity principle. Concretely, such a winning committee would grant each party a number of seats proportional to its relative popular support, that is to say, the common relative grade it receives. Finally, we will consider that a GBC choice rule F is unanimous if, when facing such kind of profiles, it returns a winning committee W in which the proportion of seats won by each party is equal to the unanimous relative grade they each receive.

Axiom 3: Unanimity. A GBC choice rule F is unanimous if for every profile M such that:

$$M_i = M_j \quad \forall (i,j) \in [n]^2$$

F returns a committee implying, for every party r:

$$\frac{s_r(F(M))}{k} = h_i(r, M)$$

1.1.4 Monotonicity

Usually, we consider that having more popular support should lead to having more seats, all other things being equal. With plurality voting, a voting rule is monotonic if an isolated increase in the number of votes gathered by a party does not diminish its number of seats. The increase should be isolated since the total number of seats to be allocated is fixed: if all parties register an increase in the number of votes they receive, we cannot give all of them more seats, given that providing an additional seat to a party requires to deprive another party of it. Thus, only those who gain more votes than the average should be allocated more seats, and these seats should be taken to the parties with weaker increases in votes. This is why monotonicity applies only when facing isolated changes. Within the grade setting, we care about increases in the grades given to each party. If a party receives a higher grade from each voter, all other things being equal, monotonicity should state that its number of seats cannot decrease. But if some voters gives higher grades to all parties, then monotonicity does not apply anymore. It is thus preferable to consider relative grades. Indeed, if a party receives a higher relative grade from each voter, we can consider that this party is more relatively preferred (or less relatively unpreferred) to the other parties than before. With such a dynamic, it is quite natural to assume that the number of seats won by that party within the new profile must be at least equal to the initial one. Therefore, the increase in absolute grades does not need to be isolated for applying monotonicity, it is sufficient that all the relative grades increase. Further, all relative grades increase is sufficient when aiming at verifying monotonicity but it is not necessary too. Indeed, it is enough to consider the sum of relative grades gathered by a party r, denoted $H(r, M)$. Before anything else, let precise that the global sum of these grades, denoted $H(M)$, is fixed and equal to the number of voters whatever the voters' preferences: $H(M) = n$. Thus, when $H(r, M)$ increases, we can deduce that the party r is globally and relatively more preferred than initially, and hence, it should be granted at least the same number of seats than before.

Axiom 4: Monotonicity. A GBC choice rule F is monotonic if for any profiles (M, M') and for any party r such that:

$$H(r, M') \geq H(r, M)$$

F returns committees implying:

$$s_r(F(M')) \geq s_r(F(M))$$

1.2 Pareto Axioms

Then, we suggest to extend the Weak and Strong Pareto axioms to the multi-winner grade setting. Both axioms states that if we can observe a common hierarchy between a couple of alternatives established by individuals, then this hierarchy should be respected by the social preference or social choice. Regarding our case, these axioms should state that if all voters give higher grades to a party than another one, then this party should be granted a higher number of seats. If it was not the case, we could take seats from the low-graded party and give them to the high-graded one while giving more representativeness to all voters. Thus, these axioms ensure that there is no alternative committee giving more representativeness to all voters simultaneously.

1.2.1 Weak Pareto Axiom

When considering the Weak Pareto axiom, the common hierarchy is not strict. If each voter gives a grade to a party r that is at least as good as the one she gives to a party t, the party r should be granted a number of seats at least as high as that of party t.

Axiom 5: Weak Pareto Axiom. A GBC choice rule F is weakly Paretian if for any profile M and any couple of parties (r, t) such that:

$$g_i(r, M) \succcurlyeq g_i(t, M) \quad \forall i \in [n]$$

F returns a committee implying:

$$s_r(F(M)) \geq s_t(F(M))$$

1.2.2 Strong Pareto Axiom

Then, considering the Strong version of the Pareto axiom, the common hierarchy is strict to the extent that there is at least one voter who gives a strictly higher grade to the commonly high-graded party. In that case, it could be desirable that this party gets a strictly higher number of seats than the other party.

Axiom 6: Strong Pareto Axiom. A GBC choice rule F is strongly Paretian if for any profile M and any couple of parties (r, t) such that:

$$g_i(r, M) \succcurlyeq g_i(t, M) \quad \forall i \in [n]$$

and

$$\exists j \in [n] \quad s.t. \quad g_j(r, M) \succ g_j(t, M)$$

F returns a committee implying:

$$s_r(F(M)) > s_t(F(M))$$

1.3 Non-Dictatorship

It is preferrable that the GBC choice rule takes into account the preferences of all voters, and not merely mimic those of a single voter. We talk about dictatorship when the social choice or social preference is only determined by one individual, and where all others have no influence on the social outcome. Thus, we want F to be non-dictatorial. In other words, the GBC choice rule F should not verify the dictatorship axiom. Applied to our setting, it would mean that there is no voter such that the winning committee W generated by F allocates seats to the parties proportionally to that voter's preferences. Precisely, the allocation of seats mirrors the preferences of a voter v_i if and only if this allocation is equally structured as her expressed preferences: the proportion of seats obtained by each party is exactly equal to its relative grade from v_i.

Axiom 7: Dictatorship. A GBC choice rule F is dictatorial if it exists a voter $v_i \in V$ such that, for every profile M and for every party r:

$$\frac{s_r(F(M))}{k} = h_i(r, M)$$

1.4 Independence of Irrelevant Alternatives in Relative Grading

From now on, we suggest to extend the famous Independence of Irrelevant Alternatives (IIA) as defined by the American economist [Arrow, 1951]. This axiom states that, when facing ranked ballots, the social ordering between two alternatives should only depend on their individual orderings. In their book, [Balinski and Laraki, 2010] already extended this principle to the single-winner MJ setting, and they called it the Independence of Irrelevant Alternatives in Grading (IIAG). This extended axiom states that the majority-grade assigned to any candidate should depend on her grades only. Our aim is to extend the IIA to the multi-winner MJ setting. Should the number of seats won by a party depend on the grades it received only? It seems that it is not the case. Since the size of the committee is fixed, it is obvious that if some parties get more votes than initially, the fact of giving them more seats necessarily implies to decrease the number of seats detained by the other parties. For being a fixed-value allocation game, it is totally conceivable that the outcome of each party also depends on the grades received by other parties. Therefore, the number of seats granted to any party may depend on the set of all grades, and not only their respective grades. However, the IIA becomes desirable when regarding relative grades. Indeed, the relative grades already contain the information about how much a party is relatively preferred to the others by any voter. It follows the same reasoning as when extending monotonicity to our setting. Thus, it is reasonable to suppose that the number of seats granted to a party should only depend on its relative grades. Consider the r^{th} party. Imagine that another party, say the t^{th} party, registers an increase in its relative grades (or rather, an increase in the sum of its relative grades). It means that the t^{th} party is globally and relatively more preferred than initially. By monotonicity, it should get more seats. If this gain in seats for the t^{th} party affects the r^{th} party, it means that the increase in $H(t, M)$ is partially or integrally compensated by a decrease in $H(r, M)$, and hence, relative grades assigned to the r^{th} party have dropped. If this gain in seats does not affect the r^{th} party, it means that the increase in $H(t, M)$ is compensated by other parties' decreases and that $H(r, M)$ remains unchanged. In all cases, $H(r, M)$

contains the necessary information to decide how much seats the r^{th} party should be given. Finally, IIAG does not suit the multi-winner MJ setting – it is not reasonable to assume that each party's number of seats depends only on their own absolute grades – but the Independence of Irrelevant Alternatives in Relative Grading (IIARG) does.

Axiom 8: Independence of Irrelevant Alternatives in Relative Grading. A GBC choice rule F is independent of irrelevant alternatives in relative grading if for any profiles (M, M') and for any party r such that:

$$h_i(r, M') = h_i(r, M) \quad \forall i \in [n]$$

F returns committees in which:

$$s_r(F(M')) = s_r(F(M))$$

2 Extending Proportionality

When a parliament is proportionally elected, it means that each party has obtained a number of seats that is proportional to the number of votes it received. It is the classical definition of proportionality. For instance, if a party gathered 35% of votes, it should be granted the same percentage of seats, when possible. More generally, if there are k seats to be filled, n voters and n_r of them voting for the r^{th} party, then this party should be granted $(\frac{n_r}{n})k$ seats, when this number is a natural integer. As a result, if a voter has a very specific opinion and is the only voter of a party ($n_r = 1$), this party should be given $(\frac{k}{n})$ seats. Thus, this ratio is the number of seats that should represent one voter, when proportionality is perfectly respected. Precisely, it is the political power that each voter actually exerts on the parliament when it is proportional, and this power is the same for all individuals. Proportionality is desirable since it finally guarantees that each voter has the same amount of influence on the elected parliament. It is egalitarian in terms of political power.

Plurality voting is perfectly appropriate when we want to implement proportionality, given that each voter can vote for one and only one party. On the contrary, approval voting and score voting are less suitable since each voter may support several parties. Thus, the classical definition of proportionality is not general enough and should be extended to these forms of voting. First, we will extend it for approval ballots, and then, for score ballots. Obviously, all notions that stand for score voting can be easily extended to grade voting (MJ voting) using the conversion function φ. Furthermore, let precise that we will begin to study the case of party-list elections, since it is more appropriated to proportionality notions, and we will then extend our results to the case of free-candidate elections, where political options are independent candidates.

2.1 Extending Proportionality to Approval Voting

2.1.1 Traditional Proportional Apportionment's failure

By applying the classical definition of proportionality to approval voting, each party should be granted $(\frac{n_r}{N})k$ seats, where N refers to the total number of votes, which is not necessarily equal to the number of voters n, as each voter can vote several times. Furthermore, a voter who approves x parties will be counted x times in N and, more precisely, will be counted once in x electorates n_r. Thus, with R parties, we have

$$N = \sum_{r=1}^{R} n_r \geq n$$

Let consider the following approval profile under its ballot form, with $n = 8$, $R = 3$ and $k = 300$.

P_1	1	1	1	1	1	0	1	0	6
P_2	0	0	1	1	1	1	0	1	5
P_3	0	1	0	0	1	0	1	1	4
	v_1	v_2	v_3	v_4	v_5	v_6	v_7	v_8	Total

The first party gets six votes, the second gets five and the third gets four. Thus, $N = 15$. We denote by $\mu_r = (\frac{n_r}{N})$ the proportion of votes the r^{th} party received, and s_r the number of seats that should be theoretically granted to this party.

- $\mu_1 = (\frac{n_1}{N}) = \frac{6}{15}$ and then $s_1 = (\frac{n_1}{N})k = (\frac{6}{15})300 = 120$
- $\mu_2 = (\frac{n_2}{N}) = \frac{5}{15}$ and then $s_2 = (\frac{n_2}{N})k = (\frac{5}{15})300 = 100$
- $\mu_3 = (\frac{n_3}{N}) = \frac{4}{15}$ and then $s_3 = (\frac{n_3}{N})k = (\frac{4}{15})300 = 80$

Now, we seek to measure the political influence that is exerted by each voter on the elected parliament. A voter is represented by parliamentarians belonging to a party she approved. However, a parliamentarian may represent more than one voter. Therefore, for each political group in the elected assembly (in our example, there are three groups), we propose to share the number of seats among their supporters in order to measure the political power each of these supporters derive from this party only. Following the same reasoning, we can define the amount of political power of a voter by the weighted sum of the number of seats won by approved parties, where the weight is the proportion her approval ballots represent among the total of ballots perceived by the considered party

Proposition 1. Considering approval voting, if p_i refers to the political power exerted by any voter $v_i \in V$ on the elected parliament W, then p_i is computed as follows:

$$p_i = \sum_{r \in T_i} \left(\frac{1}{n_r}\right) s_r(W)$$

Where $T_i = \{r \in [R], P_r \subset A_i\}$

When the traditional proportional allocation of seats is conducted, that is to say, when $s_r = (\frac{n_r}{N})k$ for every party r, the measure of the political influence becomes:

$$p_i = \sum_{r \in T_i} \left(\frac{1}{n_r}\right)\left(\frac{n_r}{N}\right)k = \sum_{r \in T_i} \left(\frac{k}{N}\right) = |T_i|\left(\frac{k}{N}\right)$$

For plurality voting, such an allocation should display a p_i equal to $(\frac{k}{n})$ for all voters. Yet, plurality voting implies $|T_i| = 1$ for every voter v_i and $N = n$ since each voter can vote for exactly one party. These results actually imply that $p_i = (\frac{k}{n})$ for every voter v_i. Therefore, this formulation of the political power is consistent with our introductory analysis.

Precisely, we choose such a formulation since we cannot say that the political power of a voter is the mere sum of the number of seats given to the parties she approves, given that each party's seats are supposed to represented several voters. Therefore, for each party's seats, our first intuition is to allocate them uniformly between their supporters. The uniformity is obviously implied by the fact that each voter represents exactly the same proportion of an electorate she belongs to ($\frac{1}{n_r}$ for the electorate of the r^{th} party). Nevertheless, since voters may approve different numbers of parties, they do not necessarily represent the same proportion of the total number of votes N. This is why voters do not necessarily have the same amount of political power. In our example, voter v_1 approves

one party whereas voter v_5 approves all of them. We thus predict that v_5 has a stronger influence on the parliament than v_1. Using the previous political power formulation, we can confirm our prediction:

$$p_1 = |T_1|\left(\frac{k}{N}\right) = 1\left(\frac{300}{15}\right) = 20 < p_5 = |T_5|\left(\frac{k}{N}\right) = 3\left(\frac{300}{15}\right) = 60$$

More generally, if a voter approves δ more parties than another voter, she will have a political power surplus of $\delta(\frac{k}{N})$ relatively to that voter. The simple fact of approving one more party provides $(\frac{k}{N})$ supplementary units of political power. To be honest, this claim is true in a static analysis, where N is constant even if voters change their votes. However, in a dynamic analysis, the value of N must change when a voter modifies her ballot paper. If a voter approves one more party, the total number of votes N will increase by one unit. Consequently, the political power gain of approving y more parties, all other things being equal, is computed as follows (Appendix A):

$$\Delta_y p_i = p_i(x_i + y, N + y) - p_i(x_i, N) = (x_i + y)\left(\frac{k}{N+y}\right) - x_i\left(\frac{k}{N}\right) = \left(\frac{y}{N+y}\right)\left(\frac{N - x_i}{N}\right)k$$

Where $x_i = |T_i|$. This result is valid for a discrete analysis. For a continuous analysis, the marginal political power can be found by differentiating p_i with respect to x_i without forgetting that N is strictly increasing with x_i.

$$p_i(x_i) = x_i\left(\frac{k}{N(x_i)}\right) = x_i\left(\frac{k}{\sum_{j=1}^{n} x_j}\right)$$

$$\frac{\partial p_i}{\partial x_i} = k\frac{(\sum_{j=1}^{n} x_j) - x_i}{(\sum_{j=1}^{n} x_j)^2} = k\left(\frac{N - x_i}{N^2}\right)$$

We remark that the marginal political power is decreasing with the number of approved parties:

$$\frac{\partial^2 p_i}{\partial x_i^2} = -\frac{k}{N^2} < 0$$

As a result, the more a voter approves parties, the more she will be granted political representation, but less proportionality than the increase in approved parties. There is a saturation effect.

Finally, we proved that, when considering approval voting, the traditional method of proportional apportionment can violate the concept of proportionality, where each individual should have exactly the same amount of influence on the elected parliament. Hence, it is necessary to revise the way we define a proportional apportionment in order that each voter derive exactly the same political power from the winning assembly.

2.1.2 The Voting Gauge sharing

Each voter must have the same weight in the total number of votes but, in the same time, each voter must be able to approve more than one party. A solution is to grant each individual the same number of approval votes. For example, each voter would be obligated to vote for three parties, not less, not more. However, with such a system, we are not in the free approval voting anymore and we lose in generality. Another solution is to give each voter a similar voting gauge that can be shared among the approved parties. For instance,

with a voting gauge equal to one, a voter who approves three parties will give one third of her gauge to each of them. The sharing is supposed to be uniform as approval ballots does not enable voters to specify their preferences among the parties they approve. More generally, the value of the approval ballot belonging to a voter approving x parties should be equal to $(\frac{1}{x})$, in order that the sum of her votes be equal to one whatever the number of parties she approved.

Coming back to our example, the approval profile can be thus represented as follows:

P_1	1	1/2	1/2	1/2	1/3	0	1/2	0	20/6
P_2	0	0	1/2	1/2	1/3	1	0	1/2	17/6
P_3	0	1/2	0	0	1/3	0	1/2	1/2	11/6
	v_1	v_2	v_3	v_4	v_5	v_6	v_7	v_8	Total

Since each voter has a voting gauge equal to one, we have $N = n$. If the common voting gauge was equal to θ, we would have had $N = \theta n$. Then, we apply the proportional apportionment of seats, as if there were 20/6 people who vote for the first party, 17/6 for the second party etc.

- $\mu_1 = (\frac{20/6}{8})$ and then $s_1 = (\frac{20/6}{8})300 = 125$ (+5)
- $\mu_2 = (\frac{17/6}{8})$ and then $s_2 = (\frac{17/6}{8})300 = 106,25 \approx 106$ (+6)
- $\mu_3 = (\frac{11/6}{8})$ and then $s_3 = (\frac{11/6}{8})300 = 68,75 \approx 69$ (-11)

A party that loses seats relatively to the previous allocation is a party with an electorate where votes were less weighted than the average, and hence, an electorate composed of voters who tend to approve more parties than the others.

If we denote $q_i(r)$ the dummy variable representing the approving decision of voter v_i with respect to the r^{th} party with $q_i(r) = 1$ when this party is approved and when $q_i(r) = 0$ when it is disapproved, so $Q(r)$, the total of voting points gathered by the r^{th} party, should be equal to:

$$Q(r) = \sum_{i=1}^{n} \left(\frac{q_i(r)}{x_i} \right)$$

Theoretically, the number of seats granted to that party is then equal to:

$$s_r = \left(\frac{Q(r)}{n} \right) k = \frac{k}{n} \sum_{i=1}^{n} \left(\frac{q_i(r)}{x_i} \right)$$

With such a mechanism, all voters should have the same political power. The previous formulation of that power is now obsolete, since each voter does not necessarily make up the same proportion of a given electorate. Indeed, in our example, among votes perceived by the first party, voter v_1 provides one vote, v_3 provides one half of vote and v_5 one third. Therefore, in order to compute the weight of a voter within an electorate, we should divide the voting points she gives to the considered party by the total of voting points gathered by that party.

Proposition 2. Considering approval voting and applying voting gauge sharing, if $\tilde{p}_i$ refers to the political power exerted by any voter $v_i \in V$ on the elected parliament W, then $\tilde{p}_i$ is computed as follows:

$$\tilde{p}_i = \sum_{r \in T_i} \left(\frac{1/x_i}{Q(r)}\right) s_r(W)$$

Applying the traditional proportional allocation, leading to $s_r = (\frac{Q(r)}{n})k$, we obtain:

$$\tilde{p}_i = \sum_{r \in T_i} \left(\frac{1/x_i}{Q(r)}\right)\left(\frac{Q(r)}{n}\right)k = \frac{k}{n} \sum_{r \in T_i} \left(\frac{1}{x_i}\right) = \frac{k}{n}$$

As expected, each voter exerts the same amount of power on the elected assembly. Proportionality is hence respected and approval voting has not been restricted. Yet, the fact that the weight granted to a vote decreases when approving more party may be problematic, as if approving a supplementary party systematically implies that our preferences for the initially chosen parties were less strong than before, since we give them fewer voting points. Imagine two voters, v_1 and v_2, the former approving P_1, and the second P_1 and P_3, with exactly the same degree of support. Precisely, if the first voter supports 90% of the ideas defended by the first party, then the second voter also supports 90% of the ideas defended by P_1 and P_3 respectively. Whereas v_1 gives one entire vote to P_1 for such an approbation rate, v_2 only provides one half to each approved parties with an identical approbation rate. Consequently, the voting gauge rule do not permit to reflect perfectly voters' preferences. Furthermore, the uniform sharing does not permit to express differences between approved parties. But to be honest, these are approval ballots that underlie this inability. Fortunately, with grade ballots, preferences could be better expressed and voting gauge sharing could be accordingly improved.

2.2 Extending Proportionality to Grade Voting

When it comes to grade voting, traditional proportional apportionment also leads to a break of the equal power principle. Indeed, each voter assigns a grade to each party, and we can consider that the total of voting points gathered by a party is the sum of the grades it receives. Then, giving each party a number of seats proportional to their voting points will generate inequalities in terms of political power; a voter who gives high grades on average will make up a higher proportion of the global voting points than another who gives lower grades, and hence she will have more political influence. This is the same distortion mechanism than the one faced when studying approval voting. And the intuitive solution is quite the same: each voter should be granted a voting gauge that will be shared among the several parties with respect to the grades they were given.

Let denote $g_i(r)$ the grade that voter v_i gives to the r^{th} party. The voting points given by this voter to that party depend on the relative position of $g_i(r)$ with respect to the other grades she assigns. If we normalize the voting gauge to one, then voting points granted to a party by this voter are equal to the grade this party receives from her, divided by the sum of the grades given by this voter. Finally, such a definition corresponds to

the relative grade $h_i(r)$. With such a counting system, the sum of voting points given by a voter is necessarily equal to one, and thus equal power principle should be respected. As a result, the global sum of voting points is equal to n. By denoting $H(r)$ the sum of voting points gathered by the r^{th} party, the number of seats theoretically granted to the r^{th} party after traditional proportional apportionment is:

$$s_r = \left(\frac{H(r)}{n}\right)k = \left(\frac{k}{n}\right)\sum_{i=1}^{n}\frac{\varphi(g_i(r))}{\sum_{t=1}^{R}\varphi(g_i(t))}$$

By taking a voting gauge equal to θ, the number of seats per party would be exactly the same. We would have $H(r)$ the sum of relative grades, each of them multiplied by θ, but when determining s_r, we would have to divide by θn instead of n. Therefore, whatever the value taken by θ, all voting gauge methods give the same winning committee W, for a given profile M.

Election rule 1: Voting Gauge Methods (VGM). Let $F^{\theta VG}$ the GBC choice rule that implements a voting gauge equal to θ. Thus, for any profile M, $F^{\theta VG}$ returns a winning committee such that, for every party r:

$$s_r(F^{\theta VG}(M)) = \left(\frac{k}{\theta n}\right)\sum_{i=1}^{n}\theta\frac{\varphi(g_i(r))}{\sum_{t=1}^{R}\varphi(g_i(t))}$$

Again, we have to redefine the way we formulate the political power. For a given voter, it always depends on the number of seats granted to each party and the proportion she represents within each electorate. Nevertheless, all parties must now be taken into account for every voter, given that each voter is represented by all the parties but at different levels. Intuitively, it seems reasonable to weight each party's seats by the grade they were given but the weighting is already operated through the proportion term: when a voter assigns a higher grade to a party, the proportion of the voting points she gives to that party is also higher and that party's seats are most weighted.

Proposition 3. Considering grade voting and applying voting gauge sharing, if $\hat{p}_i$ refers to the political power exerted by any voter $v_i \in V$ on the elected parliament W, then $\hat{p}_i$ is computed as follows:

$$\hat{p}_i = \sum_{r=1}^{R}\left(\frac{h_i(r)}{H(r)}\right)s_r(W)$$

By introducing the theoretical value of s_r that is obtained with a proportional allocation, we should find that $\hat{p}_i$ is always equal to the size of the parliament k divided by the number of voters n.

$$\hat{p}_i = \sum_{r=1}^{R}\left(\frac{h_i(r)}{H(r)}\right)\left(\frac{H(r)}{n}\right)k = \left(\frac{k}{n}\right)\sum_{r=1}^{R}h_i(r) = \left(\frac{k}{n}\right)$$

This result is trivial since each voter is granted exactly the same voting gauge, and cannot provide more voting points whatever the grades she gives.

Let consider the following MJ profile, where grades vary discretely between 0 and 9 by one unit, with $n = 8$, $R = 3$ and $k = 200$.

P_1	6	5	4	1	2	0	4	0	22
P_2	3	2	1	9	6	2	2	3	28
P_3	1	8	0	0	2	3	9	7	30
	v_1	v_2	v_3	v_4	v_5	v_6	v_7	v_8	Total

The global sum of grades is equal to $G = 80$. The first party receives 27.5% of votes, the second party receives 35% and the third 37.5%. By applying traditional proportional apportionment, 55 seats are won by the first party, 70 seats by the second and 75 seats by the third. Do voters have the same influence on the elected parliament?

Consider voters v_6 and v_7. The former is one of those who assign the lowest grades on average, with a mean of $\bar{g_6} = 5/3$, whereas the latter is one of those who assign the highest grades with a mean of $\bar{g_7} = 5$. We expect v_7 to have more political power than v_6.

$$p_6 = \left(\frac{0}{22}\right)55 + \left(\frac{2}{28}\right)70 + \left(\frac{3}{30}\right)75 = \left(\frac{25}{2}\right)$$

$$p_7 = \left(\frac{4}{22}\right)55 + \left(\frac{2}{28}\right)70 + \left(\frac{9}{30}\right)75 = \left(\frac{75}{2}\right)$$

The seventh voter has three times more political power than the sixth voter. This relation is predictable since $\bar{g_7} = 3\bar{g_6}$, meaning that, on average, v_7 assigns grades three times as high as those assigned by v_6, and hence, the quantity of voting points provided by v_7 is three times as big as those provided by v_6. Consequently, the proportion that v_7's voting points represents in the global sum of voting points G is equal to the triple of that of v_6's voting points.

Now, we give each voter a voting gauge normalized to one. The MJ profile can be transformed as follows:

P_1	6/10	5/15	4/5	1/10	2/10	0/5	4/15	0/10	69/30
P_2	3/10	2/15	1/5	9/10	6/10	2/5	2/15	3/10	89/30
P_3	1/10	8/15	0/5	0/10	2/10	3/5	9/15	7/10	82/30
	v_1	v_2	v_3	v_4	v_5	v_6	v_7	v_8	Total

Obviously, $G = n$. We immediately remark that it is now P_2 that gathers the more voting points, and thus will have the highest number of seats. P_1 has perceived 28,75% of votes (57.5 seats), P_2 has gathered approximatively 37,08% (74,16 seats) and P_3 is close to 34,17% (68.34 seats). We decide to allocate seats using the largest remainder method (Hare quota method).

Therefore, P_1 directly wins 57 seats, P_2 gets 74 seats and 68 seats are given to P_3. There is still one seat to be allocated. It will be granted to P_1 for having the largest remainder. The elected assembly is then not perfectly proportional but it is the least unproportional we can do. We expect political power of voters to be very close to each

other. For instance, for voters v_6 and v_7, who were previously displaying uneven amounts of political influence, we now have:

$$\hat{p}_6 = \left(\frac{0/5}{69/30}\right)58 + \left(\frac{2/5}{89/30}\right)74 + \left(\frac{3/5}{82/30}\right)68 \approx 24.9$$

$$\hat{p}_7 = \left(\frac{4/15}{69/30}\right)58 + \left(\frac{2/15}{89/30}\right)74 + \left(\frac{9/15}{82/30}\right)68 \approx 24.98$$

Normally, $\hat{p}_i$ should be equal to $(\frac{k}{n}) = \frac{200}{8} = 25$. The values computed for voters v_6 and v_7 are very close to this ideal. If we had kept the theoretical values for the allocation of seats, as if seats were perfectly divisible, we would have found $\hat{p}_i = 25 \; \forall i \in [n]$. The inequality $\hat{p}_6 < 25$ can be explained by the fact that we give one more seat to the first party whereas it was weighted by a grade of zero from this voter.

Finally, we show that it is necessary to distinguish the proportional allocation from the concept of proportionality, the former being a mechanism and the latter being a principle, and that a proportional allocation does not systematically generate an assembly that respects such a principle. Right now, such a principle will be called Equal power principle.

Axiom 9: Equal power principle (Epp). If $p_i(W, M)$ refers to the amount of political power exerted by any voter $v_i \in V$ on the elected parliament W when considering the profile M, then a GBC choice rule F respects the equal power principle if, for any profile M,

$$p_i(F(M), M) = \frac{k}{n} \quad \forall i \in [n]$$

If voters do not make up the same proportion of the total of votes, or more generally, the total of voting points, a proportional allocation will lead to a violation of equal power principle. Conversely, if voters exactly make up the same proportion of the global sum of voting points, a proportional allocation will generate a proportional parliament.

Theorem 1. Let N the total of voting points and $\Sigma g_i(M)$ the sum of voting points provided by any voter $v_i \in V$ for every profile M. Also consider $\tilde{F}$ the proportional allocation rule.

- If $\exists(i, j)$ such that $\Sigma g_i(M) > \Sigma g_j(M)$, then $p_i(F(M), M) > p_j(F(M), M)$ and Epp is not verified.
- If $\Sigma g_i(M) = \Sigma g_j(M) \; \forall(i, j)$, then $p_i(F(M), M) = \frac{k}{n} \; \forall i \in [n]$ and Epp is verified.

2.3 Generalizing to Free-Candidates Elections

Until now, we worked on elections structured by political parties. Thus, when a party was granted a certain number of seats, it was able to fill all of them. Indeed, when there is a party-list election, each party suggests an ordered list of candidates, in which there are as many candidates as seats to be filled. Nevertheless, we want to extend our previous results to free-candidate elections, where there are no political structures and where each voter has to give her opinion on each candidate separately.

Let consider the following approval profile A_0 under its ballot form with $m = 6$, $n = 8$, $k = 4$.

c_1	1	0	1	0	0	1	0	1
c_2	1	1	0	1	0	1	0	1
c_3	0	1	1	1	0	0	1	0
c_4	1	0	1	0	1	0	1	1
c_5	0	0	1	0	0	0	0	0
c_6	0	0	0	1	0	1	0	1
	v_1	v_2	v_3	v_4	v_5	v_6	v_7	v_8

We decide to give each voter a voting gauge equal to one, that will be shared uniformly between her approval votes. Therefore, the approval profile can be rewritten as follows:

c_1	1/3	0	1/4	0	0	1/3	0	1/4	14/12	15%
c_2	1/3	1/2	0	1/3	0	1/3	0	1/4	21/12	22%
c_3	0	1/2	1/4	1/3	0	0	1/2	0	19/12	20%
c_4	1/3	0	1/4	0	1	0	1/2	1/4	28/12	29%
c_5	0	0	1/4	0	0	0	0	0	3/12	3%
c_6	0	0	0	1/3	0	1/3	0	1/4	11/12	11%
	v_1	v_2	v_3	v_4	v_5	v_6	v_7	v_8	Total	Prop.

If c_4 was a political party, it should be granted 29% of seats approximately. However, c_4 is a unique candidate, we cannot decompose it into many elements as we did for parties, and hence, we cannot give her several seats. One candidate can be granted either one seat, or zero seat. There are no alternatives. This binarity is problematic: if we cannot give to a candidate the number of seats she proportionally deserves, we will never be able to generate a proportional assembly, and the equal power principle will be impossible to verify. Except in some very rare cases. For instance, if in our application four candidates had gathered 25% of voting points and the two others 0% of voting points, each of the four candidates would have won one seat and the elected parliament would have been perfectly proportional. However, what should we do in the general configuration?

2.3.1 The Iterated Normalization Process

A first solution is to normalize to one the highest number of seats that should theoretically be given. Precisely, we give only one seat to the candidate who arrives in the first position, instead of giving her $(\frac{n_{max}}{n})k$ seats, and we apply the same normalization process to the other candidates. As a result, another candidate c_j is now entitled to $(\frac{n_j}{n_{max}})$ seats. In our example, such a process engenders:

	Voting points	Proportion	Theoretical seats	Normalized seats
c_1	14/12	15%	0.6	15/29
c_2	21/12	22%	0.88	22/29
c_3	19/12	20%	0.8	20/29
c_4	28/12	29%	1.16	1
c_5	3/12	3%	0.12	3/29
c_6	11/12	11%	0.44	11/29

Given that c_4 should theoretically be given 1.16 seat, but we can only give her one seat, we consider that winning 1.16 seat give the right to have one seat, and this conversion process must be applied for all candidates in order to respect proportionality. Generally, the problem before the normalization is that we cannot give candidates more than one seat. After the normalization, this problem systematically disappears but another issue arises: we cannot give candidates pieces of one seat, since seats are not perfectly divisible. With such a process, none of the other candidates deserve to have one seat, except if ties occur at the first place.

After giving the best candidate one seat when we were to allocate k seats between m candidates, we still have to allocate $(k-1)$ vacant seats between the $(m-1)$ remaining candidates. Intuitively, we suggest to reiterate the normalization process that we use for k seats and m candidates to the $(k-1)$ vacant seats and the $(m-1)$ remaining candidates, as if there were a new election with new parameters. This method is not equivalent to give the $(k-1)$ seats to the $(k-1)$ best remaining candidates, and we will prove it. Obviously, we will not ask voters to express their preferences on the remaining candidates again, given that this information is already contained in the approval ballots, and that removing one candidate from the initial casting should not modify their approbation choice regarding the other candidates. Indeed, with approval voting, expressed preferences are supposed to be cardinal, that is to say, each voter is supposed to say if she approves or not each candidate in herself and not by making comparison with other candidates. On the contrary, with ranked ballots, where each voter is asked to rank the several candidates, expressed preferences are ordinal: we are able to know if a voter prefers one candidate to another, but we cannot know if she likes these candidates or not. Coming back to our example, we propose to truncate the previous approval profile by removing c_4 and we obtain:

c_1	1	0	1	0	0	1	0	1
c_2	1	1	0	1	0	1	0	1
c_3	0	1	1	1	0	0	1	0
c_5	0	0	1	0	0	0	0	0
c_6	0	0	0	1	0	1	0	1
	v_1	v_2	v_3	v_4	v_5	v_6	v_7	v_8

Applying the voting gauge sharing now generates:

c_1	1/2	0	1/3	0	0	1/3	0	1/3	9/6	21%
c_2	1/2	1/2	0	1/3	0	1/3	0	1/3	12/6	29%
c_3	0	1/2	1/3	1/3	0	0	1	0	13/6	31%
c_5	0	0	1/3	0	0	0	0	0	2/6	5%
c_6	0	0	0	1/3	0	1/3	0	1/3	6/6	14%
	v_1	v_2	v_3	v_4	v_5	v_6	v_7	v_8	Total	Prop.

The best candidate is now c_3 and thus we give her one seat. We iterate the normalization process until all seats are filled. We remark that, in the first round, c_2 was gathering more votes than c_3. If we had decided to give the second seat to the second-best candidate as defined by the initial order, the second seat would have been given to c_2 instead of c_3. Such an observation proves that the iterated method we propose is not tantamount to the very basic method where we give the k seats to the k best candidates, since the ordering between candidates may change round after round. We also remark that voter v_5 does not contribute to the second round. A priori, excluding somebody from the election can be seen as undemocratic, but her ballot clearly specifies that she does not approve any of the remaining candidates. Yet, when a voter does not support a candidate or a party, she normally does not give them her vote. So, the fact that v_5 gives zero voting point to each of these candidates is consistent with her expressed preferences, as if this voter were going to the polls to slide a blank ballot paper.

Generalizing to grade voting, the process is exactly the same. A MJ profile where we give each voter the same voting gauge permit to say how many seats should be given to each candidate, then we apply the normalization process, permitting to give one seat to the first-ordered candidate, we remove this candidate from the initial MJ profile to obtain a truncated one, and we repeat the same process on this new profile until all seats are allocated. At each round, the same grades $g_i(c_j)$ are considered, but the relative grades $h_i(c_j)$ may potentially change. This is why we denote $h_{i,z}(c_j)$ the relative grade given by voter v_i to the j^{th} candidate at the round z, $H_z(c_j)$ the total of voting points perceived by that candidate in this round, and c^z the candidate winning the z^{th} seat. For instance, in our previous application, we can say that $c_4 = c^1$, since the fourth candidate wins the first seat. The relative grade $h_{i,z}(c_j)$ is thus equal to $g_i(c_j)$ divided by the sum of all grades provided by v_i except for grades given to already-elected candidates $\{c^1; c^2; \ldots; c^{z-1}\}$.

$$h_{i,z}(c_j) = \frac{\varphi(g_i(c_j))}{\sum_{c_\lambda \in C_z} \varphi(g_i(c_\lambda))}$$

Where C_z is the set of remaining candidates at the round z:

$$C_z = C \setminus \left(\bigcup_{j=1}^{z-1} \{c^j\} \right)$$

The candidate who wins the round z is the one who gathers the highest number of voting points:

$$c^z = arg \max_{c_j \in C_z} \left\{ \sum_{i=1}^{n} h_{i,z}(c_j) \right\}$$

Election rule 2: The Iterated Normalization Process (INP). Let F^{INP} the GBC choice rule that implements the Iterated Normalization Process. Thus, for any profile M, F^{INP} returns a winning committee such that:

$$F^{INP}(M) = \bigcup_{z=1}^{k} \left\{ arg \max_{c_j \in C_z} \left\{ \sum_{i=1}^{n} \frac{\varphi(g_i(c_j))}{\sum_{c_\lambda \in C_z} \varphi(g_i(c_\lambda))} \right\} \right\}$$

With:

$$C_z = C \setminus \left(\bigcup_{q=1}^{z-1} \left\{ arg \max_{c_j \in C_q} \left\{ \sum_{i=1}^{n} \frac{\varphi(g_i(c_j))}{\sum_{c_\lambda \in C_q} \varphi(g_i(c_\lambda))} \right\} \right\} \right)$$

2.3.2 The Least Unproportional Decision

In a context of free-candidate elections, we know that we cannot generate a perfectly proportional parliament, except in some very specific configurations. If we cannot reach perfect proportionality, we can however seek to constitute a parliament which is the closest possible to this ideal. In other words, we want to elect the least unproportional parliament. Before anything else, it is necessary to define the way we measure the disproportionality degree of any assembly. For each voter v_i, we know that there could be a gap between her actual political power p_i and its ideal value ($\frac{k}{n}$). By denoting e_i this power gap, we can state that every e_i measures a proportionality loss at the individual level. In order to compute the collective proportionality loss, that would be a measure of disproportionality for every assembly, we have to aggregate the individual proportionality losses. The aggregation process could not be a simple arithmetic sum since it would permit mutual compensation effects: if half the voters had $e_i = e$ and the others had $e_i = -e$, the collective proportionality loss would be assessed to zero, whereas the elected parliament would not be proportional at all. A solution is to consider the absolute value of power gaps in order to avoid such effects.

$$\sum_{i=1}^{n} |e_i| = \sum_{i=1}^{n} \left| p_i - \frac{k}{n} \right|$$

Furthermore, it is conceivable to consider the square of each power gap. This choice not only permits to avoid the previously mentioned effects, but it also makes a smooth gaps' sharing more desirable than an uneven one. For instance, consider there are four voters who have to choose between two unproportional parliaments, W_A and W_B. In both cases, the sum of the power gaps in absolute value is the same and equal to eight. Nevertheless, the sharing of power gaps among voters is different:

$$e_{1,A} = 2 \ ; \ e_{2,A} = -2 \ ; \ e_{3,A} = 2 \ ; \ e_{4,A} = -2$$

$$e_{1,B} = 1 \ ; \ e_{2,B} = -1 \ ; \ e_{3,B} = 3 \ ; \ e_{4,B} = -3$$

We immediately remark that the parliament W_A displays a more egalitarian gaps' sharing than W_B does. If we measure the global proportionality loss by using the sum of power gaps in absolute value, we cannot differentiate the two parliaments. If we measure it by using the sum of squares of power gaps, then the disproportionality degree of W_A is

estimated to 16 whereas that of W_B is assessed to 20. Consequently, with such a disproportionality measure, W_A is considered less unproportional than W_B. It is reasonable to be able to make such a distinction between differently gaps-shared configurations in order to respect fairness principles. Finally, we propose to assess the collective proportionality loss of every assembly W by the following formula:

$$Cpl_M(W) = \sum_{i=1}^{n} \left(p_i(W, M) - \frac{k}{n}\right)^2$$

It is now necessary to compute the political power of each voter in a context of a free-candidate election. We consider grade voting, for being the most general way of representing voters' preferences. For each voter, the determination of the political influence she exerts on the elected assembly will depend on the relative grade she gives to each elected candidate, and the proportion that grade represents among each elected candidate's voting points. We denote $\mathring{p}_i$ the amount of political power in that context.

$$\mathring{p}_i(W, M) = \sum_{c_j \in W} \frac{h_i(c_j)}{H(c_j)}$$

Our aim is to choose the parliament with the lowest disproportionality level, that is to say, the parliament W which minimizes the following function:

$$Cpl_M(W) = \sum_{i=1}^{n} \left(\frac{k}{n} - \sum_{c_j \in W} \frac{h_i(c_j)}{H(c_j)}\right)^2$$

This method, that will be called the Least Unproportional Decision (LUD), not only permits to determine a winning parliament, but it also enables to rank all possible parliaments, by giving a measure of their disproportional degree. Thus, the LUD is a GBC ranking rule, for which it exists an associated GBC choice rule denoted F^{LUD}.

Election rule 3: The Least Unproportional Decision (LUD). Let F^{LUD} the GBC choice rule that chooses the least disproportional committee. Thus, for any profile M, F^{LUD} returns a winning committee such that:

$$F^{LUD}(M) = arg \min_{W \subset C} \left\{ \sum_{i=1}^{n} \left(\frac{k}{n} - \sum_{c_j \in W} \frac{h_i(c_j)}{H(c_j)}\right)^2 \right\}$$

Taking back the previous example, where there are $\binom{6}{4} = 15$ possible committees, we can measure their respective proportionality degree and thus determine a winning committee. According to (Appendix B), we found that the minimal $Cpl_{A_0}(W)$ is nearly equal to 0.04 and is reached for $W = \{c_1, c_2, c_3, c_4\}$. Therefore, we can write:

$$F^{LUD}(A_0) = \{c_1, c_2, c_3, c_4\}$$

3 Welfarist Approach

In their very recent article, [Peters and Skowron, 2019] opposed two kinds of proportionality respectively defended by two Nordic mathematicians in the 19th century. Precisely, they opposed proportionality in terms of political influence, defended by [Phragmèn, 1894], against proportionality in terms of welfare, defended by [Thiele, 1895]. The first type has already been studied through our previous analysis. The second type corresponds to a welfarist approach, where we care about voters' utility level [Sen, 1979]. What could be the satisfaction derived by a voter from an elected committee W? For approval voting, we easily imagine that a voter derives more utility when is elected an approved candidate than a disapproved one. For score voting, being represented by a high-scored committee member is more satisfactory than being represented by a low-scored one. It is easy to realize that the individual utility level, denoted U_i for voter v_i, really differs from the individual political power p_i. Measuring the influence a voter could have on the elected committee is not the same than measuring the welfare she derives from it. For example, in the approval setting with free candidates, when a voter v_i approves three committee members $\{c_1, c_2, c_3\}$, political power derived from this representation is the weighted sum of these seats, where the weight assigned to each candidate is the contribution of v_i in their respective election, whereas we can imagine that the voter's satisfaction could be the mere sum of these seats. Precisely, being represented by one committee member provides the same level of welfare whatever the number of voters this member represents. The individual welfare only depends on how much a voter supports each elected candidate. Finally, we saw that proportionality in terms of political influence was tantamount to equality of political powers for all voters, and we now state that, symmetrically, proportionality in terms of welfare is tantamount to equality of utility levels for all of them. Nevertheless, whereas we were able to determine an ideal value for political power, equal to $(\frac{k}{n})$, there is no ideal to reach when studying individual satisfaction. The fundamental aim is to equalize all voters' satisfaction.

Axiom 10: Equal utility principle (Eup). If $U_i(W, M_i)$ refers to the amount of utility derived by any voter $v_i \in V$ from the elected parliament W, then a GBC choice rule F respects the equal utility principle if, for any profile M, it returns a committee such that:

$$U_i(F(M), M_i) = U_j(F(M), M_j) \;\; \forall (i, j) \in [n]^2$$

However, when we face two welfare-egalitarian committees W_A and W_B, the former providing more satisfaction to each voter than the latter, we should obviously prefer W_A. It is thus reasonable to take into account the total amount of utility provided by any committee. Since welfare equity is a necessary condition, the winning committee should be the one that provides the highest collective utility among committees verifying this egalitarian condition.

This is why we will first consider the Rawlsian social welfare function. Concretely, there exist voting profiles such that it is impossible to find a committee which equalizes all voters' utility level. For example, let consider approval voting with $k = 3$, $n = 100$ and $m = 4$ where voters are equally divided into four homogeneous groups, each of this group approving a different candidate and disapproving all the others. There are $\binom{4}{3} = 4$ possible committees, in which there are always three groups of voters having one representative and one group being not represented. With such a voting profile, it is impossible to elect an equally satisfactory committee. Consequently, supposing we can measure any committee's degree of welfare inequalities, it seems that we have to select the winning committee among those displaying the lowest of that degree (which is not necessarily equal to zero). Assume we choose such a committee. Should we prefer another committee with a strongly higher collective utility level and only a few supplementary welfare inequalities? In other words, are collective utility and welfare equality substitutable goals? A positive answer enlarges the possible election rules to consider. This is why we will also study the class of Thiele's rules, and attempt to extend them to the MJ setting.

3.1 The Rawlsian social welfare function

3.1.1 The Rawlsian Committee rule

In the 70s, the American philosopher John Rawls proposed to implement the idea of equity in the determination of the social welfare [Rawls, 1971]. His work became famous and gave birth to a new way of computing the collective utility: the so-called Rawlsian social welfare function, where the social welfare is equal to the minimal individual utility level. In such a computing, we do not care about the other levels of satisfaction. We only pay attention to the most disadvantaged individual. Maximizing such a function permits to reach egalitarian situations, when possible (see the introductory example). Applying it to voting rules may be very useful. Let denote $U_i(W, M_i)$ the utility derived by voter v_i from the committee W, and R_{SW} the Rawlsian Social Welfare function such that:

$$R_{SW}(W, M) = \min_{1 \leq i \leq n} \{U_i(W, M_i)\}$$

We call Rawlsian Committee rule the GBC ranking rule that uses such a formulation of the social welfare to rank all possible committees. The associated GBC choice rule is the one that chooses W such that $R_{SW}(W, M)$ is maximized.

Election rule 4: The Rawlsian Committee rule (RC). Let F^{RC} the GBC choice rule that maximizes the Rawlsian social welfare. Thus, for any profile M, F^{RC} returns a winning committee such that:

$$F^{RC}(M) = arg \max_{W \subset C} \Big\{ \min_{1 \leq i \leq n} \{U_i(W, M_i)\} \Big\}$$

For each voter, the utility could be the sum of grades she gives to each committee members, or equivalently, the average grade she gives to them. Both options lead to the same ordering of voters, when they are hierarchized with respect to their satisfaction level, taking the average only decreases the amount of welfare displayed by their utility

function. Yet, we suppose utility functions to be ordinal. By taking the mean grade, the collective utility is then computed as follows:

$$R_{SW}(W, M) = \min_{1 \leq i \leq n} \left\{ \frac{1}{k} \sum_{c_j \in W} \varphi(g_i(c_j)) \right\}$$

Unfortunately, the Rawlsian Committee rule does not respect fundamental axioms such as monotonicity (Axiom 3). It can be showed by giving a counter example.

Suppose $n = 100$, $m = 5$ and $k = 3$. There are 50 voters who are left-wing, and 50 voters who are right-wing. Moreover, if $i < j$, then candidate c_i is on the left to c_j. Suppose that grades are integers between 0 and 10, that preferences are homogeneous within each political group, and that the profile M_0 is the following:

c_1	3	1
c_2	4	2
c_3	3	3
c_4	2	4
c_5	1	3
	Left-wing voters	Right-wing voters

There are $\binom{5}{3} = 10$ possible committees. We remark that voters have single-peaked preferences, in the sense of [Black, 1948], with respect to the linear order $c_1 > c_2 > c_3 > c_4 > c_5$, and precisely, c_2 and c_4 are the peaks of left-wing and right-wing voters respectively. Let compute the Rawlsian social welfare for each possible configuration, supposing that individual utility is the arithmetic sum of the grades given to committee members.

W	Utility of a left-wing voter	Utility of a right-wing voter	Minimal utility
$\{c_1, c_2, c_3\}$	10	6	6
$\{c_1, c_2, c_4\}$	9	7	7
$\{c_1, c_2, c_5\}$	8	6	6
$\{c_1, c_3, c_4\}$	8	8	8
$\{c_1, c_3, c_5\}$	7	7	7
$\{c_2, c_3, c_4\}$	9	9	**9**
$\{c_2, c_3, c_5\}$	8	8	8
$\{c_1, c_4, c_5\}$	6	8	6
$\{c_2, c_4, c_5\}$	7	9	7
$\{c_3, c_4, c_5\}$	6	10	6

The maximal Rawlsian social welfare is 9, and is obtained when candidates c_2, c_3 and c_4 are elected, which constitute the "centrist" committee. Thus, $F^{RC}(M_0) = \{c_2, c_3, c_4\}$. Now, assume that, after discussions, left-wing voters decide to adopt a strategy: put a 10 to their three favorite candidates, and put a 0 to the others. Their aim is clearly to elect the most left-wing committee $\{c_1, c_2, c_3\}$, in order to maximize their utility. The new profile, denoted M_1, is as follows:

c_1	10	1
c_2	10	2
c_3	10	3
c_4	0	4
c_5	0	3
	Left-wing voters	Right-wing voters

With such grade ballot papers, the Rawlsian social welfare based on expressed preferences will be modified for the three latest committees:

W	Utility of a left-wing voter	Utility of a right-wing voter	Minimal utility
$\{c_1, c_2, c_3\}$	30	6	6
$\{c_1, c_2, c_4\}$	20	7	7
$\{c_1, c_2, c_5\}$	20	6	6
$\{c_1, c_3, c_4\}$	20	8	8
$\{c_1, c_3, c_5\}$	20	7	7
$\{c_2, c_3, c_4\}$	20	9	9
$\{c_2, c_3, c_5\}$	20	8	8
$\{c_1, c_4, c_5\}$	10	8	8
$\{c_2, c_4, c_5\}$	10	9	9
$\{c_3, c_4, c_5\}$	10	10	**10**

The Rawlsian social welfare is now maximized with $W = \{c_3, c_4, c_5\}$, that is to say, the most right-wing committee. Thus, the strategy adopted by left-wing voters is, in this context, totally ineffective and even counterproductive, as it leads to electing one of their two worst possible committees, providing them an individual utility equal to 6. Through this example, we understand that the Rawlsian Committee rule is not monotonic, since c_2 has been replaced by c_5 whereas the sum of her relative grades has increased:

$$H(c_2, M_1) = 950/39 > H(c_2, M_0) = 900/39$$

3.1.2 Homogeneity of demanding natures

When defining the analytical expression of U_i, we can also imagine that individual utility is the average of the utility derived from each candidate, depending on the grade she is given:

$$U_i(W, M_i) = \frac{1}{k} \sum_{c_j \in W} u_i(g_i(c_j))$$

Where $u_i : \Omega \to \mathbb{R}$ is the grade utility function specific to voter v_i. It enables different voters to derive different utility levels from being represented by an identically-graded candidate. This formulation is very useful if we suppose that voters grant different meanings to a same grade. For instance, when $\Omega = \{$"Very Good","Good","Acceptable","Insufficient"$\}$, the lowest grade "Insufficient" can be seen as a neutral grade by a voter v_i but can be viewed as very negative by another voter v_j since it is the last grade. In that case, u_i("Insufficient") $> u_j$("Insufficient"). However, the common language should be sufficiently specified for avoiding such misinterpretations. Let recall that if [Balinski and Laraki, 2010] chose to use the adjective "common" when characterizing the set of possible grades, it is partially due to the fact that each grade is supposed to have the same meaning

for all judges (here, for all voters). Another reason could be different levels of demanding nature among voters. Even if the meaning of all the grades is commonly established, voters can derive different levels of utility from a same grade for having their own requirement degree (in French, we can talk about "niveau d'exigence"). An individual having a high demanding nature will derive less utility from a not perfect grade than an individual characterized by a lower demanding nature. For example, if v_i is less demanding than v_j, we can imagine that v_i derives more satisfaction than v_j from all grades but the highest, for which both voters derive the same amount of utility: $u_i(\alpha) = u_j(\alpha)$ and $u_i(\omega) > u_j(\omega)$ for every grade ω such that $\alpha \succ \omega$. Nevertheless, when aiming at proportionality in welfare terms, it is essential to consider that every voter derives the same satisfaction from a given grade. Otherwise, more demanding voters will be granted more consideration, and hence more seats, in order to equalize the utility levels of all voters. In that case, the Eup axiom is still verified but welfare proportionality is not, since it requires that all voters be given the same consideration (proportionality is underlain by fairness). Thus, when demanding natures are heterogeneous, the Equal utility principle is not tantamount to welfare proportionality anymore. In order that both concepts be equivalent, we will then assume that demanding natures are homogeneous, that is to say, each voter derives exactly the same utility level from an identically-graded candidate.

Assumption 1: Homogeneity of demanding natures. For any voter $v_i \in V$, the grade utility function u_i is equal to a common function $u : \Omega \to \mathbb{R}$.

This assumption must certainly not be confused with homogeneity of preferences. If preferences were homogeneous, it would mean that all voters would have the same opinion on each political alternative, and we would have $M_i = M_j$ for all $(i, j) \in [n]^2$. Finally, we assume that preferences can be heterogeneous, but not demanding natures.

3.2 Extension of Thiele's methods to the Majority Judgment setting

3.2.1 The general Thiele's rule

It is conceivable to found election rules on welfare considerations, that is to say, select the committee that maximizes the collective utility. Some researchers in multi-winner approval-based elections used this approach and designed some election rules that assign a score to each possible committee, generating a ranking between all of them. The score is supposed to be an indicator of the social welfare. The committee that gets the highest score is hence the winning committee. Among welfarist election rules, we can find Thiele's methods, a subclass of ABC ranking rules, where collective welfare is merely equal to the arithmetic sum of individual welfares, called individual scores, and where each individual score is defined as a sum of "weights" depending on the number of representatives the individual gets in the elected committee. The more representatives a voter has, the more weights she amounts and the happier she is. The utility level obtained from the j^{th} representative is denoted w_j and is the same for all voters. Furthermore, the utility of one more representative is supposed to decrease: $w_j > w_t$ for all natural integers (j, t) verifying $j < t$. Therefore, a voter who has two approved candidates in the elected committee will get a score of w_1+w_2, whereas a voter having three will obtain $w_1+w_2+w_3$.

If we want to extend Thiele's methods to a Majority Judgment analysis, we need to generalize the binary approach of the approval setting, that opposes approved candidates to disapproved ones. With a MJ ballot, a voter can give more than two grades to each candidate. As a result, for each voter, it is necessary to identify several types of candidates: those who received an excellent grade, those who received a good one, those who were rejected etc. For different types of candidates, the weights should be differentiated. Obviously, we can assume that a voter gets more utility from being represented by an excellent candidate than being represented by a poor one. Thus, the hierarchy of weights should be perfectly similar to the hierarchy of grades as defined by the common language. If we denote w_α the amount of welfare obtained by being represented by an α-graded candidate, so $\alpha \succ \beta$ should imply $w_\alpha > w_\beta$ for every pair of grades $(\alpha, \beta) \in \Omega^2$. If we aim to integrate the idea of decreasing marginal utility into the weights, as it is done in Thiele's methods, we should rather consider $w_{\alpha,j}$, the amount of welfare associated to the j^{th} member who receives a grade α. Within a same-grade group of committee members, integrating this idea implies that $w_{\alpha,j} > w_{\alpha,t}$ when $j < t$. Moreover, for a same rank j, having a better grade implies a higher weight: $w_{\alpha,j} > w_{\beta,j}$ when $\alpha \succ \beta$. To resume, for a same grade, a higher rank implies a lower utility, and for a same rank, a higher grade implies a higher utility.

We can go further assuming that a higher grade always implies a higher utility, independently of ranks: $w_{\alpha,j} > w_{\beta,t}$ when $\alpha \succ \beta$, for all natural integers (j, t). Indeed, a voter always prefers to have one more excellent representative than one more good representative, even if she has already a lot of excellent ones and a few good ones. For instance, if an elected committee provides fifty excellent members and only one good member to a voter, and if the voter would have the power to add a new member to the committee among the unelected candidates, she would choose an excellent candidate again (if possible), given that it is obviously better than adding a candidate with a lower grade. By analog means, in microeconomic analysis, it is tantamount to considering a consumer choosing between two goods, for example, chocolate and flour, with decreasing marginal utilities for each good, but where the marginal utility of chocolate is always superior to the marginal utility of flour, whatever the consumed quantities of each good, implying that the lowest marginal utility obtained from chocolate is strictly higher than the highest marginal utility obtained from flour. As a result, the consumer will only buy chocolate. When it comes to our voter preferences, it means that the weights associated to a higher grade are all superior to the weights associated to a weaker grade. For instance, when we have three grades $\{\alpha, \beta, \gamma\}$ such that $\alpha \succ \beta \succ \gamma$, our hypothesis implies that:

$$w_{\alpha,1} > \ldots > w_{\alpha,k} > w_{\beta,1} > \ldots > w_{\beta,k} > w_{\gamma,1} > \ldots > w_{\gamma,k}$$

Such a condition could be called complete grade determinacy (CGD). Given that this condition is very intuitive, we will assume that it is always verified throughout the paper. In our example, when decreasing marginal utility is already assumed, this condition can be reduced to the following inequalities: $w_{\alpha,k} > w_{\beta,1}$ and $w_{\beta,k} > w_{\gamma,1}$ given that all other inequalities are already implied by decreasing marginal utility.

Assumption 2: Complete Grade Determinacy (CGD). We say that grades are completely determinants when it comes to hierarchize the weights if, for any pair of grades $(\alpha, \beta) \in \Omega^2$ such that $\alpha \succ \beta$ and every pair of ranks $(j, t) \in [k]^2$, we have:

$$w_{\alpha,j} > w_{\beta,t}$$

In order to compute the score of a voter, it is possible to do the sum of weights within each same-grade group, and then to add up all the values obtained. Therefore, the score of voter v_i is computed as follows:

$$f_i(W, M_i) = \sum_{\alpha \in \Omega} \sum_{j=1}^{|W \cap M_i(\alpha)|} w_{\alpha,j}$$

Then, the score that will be assigned to any committee W is the arithmetic sum of such individual scores. Thus, we suggest to call Thiele's Grade Voting rule the extension of the Thiele's rule to the MJ setting. The Thiele's Grade Voting is naturally a GBC ranking rule, it assigns a score to each possible committee in order to rank them. However, from that ranking, we can derive a social choice. If we denote F^{TGV} the associated GBC choice rule, then F^{TGV} returns the committee that maximizes the global score.

Election rule 5: Thiele's Grade Voting rule (TGV). Let F^{TGV} the GBC choice rule that maximizes the global score as defined by Thiele's Grade Voting rule. Thus, for any profile M, F^{TGV} returns a winning committee such that:

$$F^{TGV}(M) = arg \max_{W \subset C} \left\{ \sum_{i=1}^{n} \left(\sum_{\alpha \in \Omega} \sum_{j=1}^{|W \cap M_i(\alpha)|} w_{\alpha,j} \right) \right\}$$

3.2.2 Weight Differences Independence

Another assumption could be weight differences independence (WDI). We define the weight difference as the relative difference between weights of different grades for a given rank or, symmetrically, between weights of different ranks for a given grade. First, when dealing with different grades, for example α and β, the weight difference is equal to $w_{\alpha,j}/w_{\beta,j}$ for a fixed rank j. It permits to assess the welfare relative gain (or loss) of being represented by an α-graded candidate instead of a β-graded one, for a same rank. Then, when focusing on different ranks, say j and t, the weight difference is equal to $w_{\alpha,j}/w_{\alpha,t}$ for the grade α, and give an idea on how the marginal utility evolves between two ranks within a same-grade group. The WDI hypothesis states that weight differences do not depend on the fixed variable. In our examples, this assumption implies that $w_{\alpha,j}/w_{\beta,j}$ is rank-invariant and that $w_{\alpha,j}/w_{\alpha,t}$ is grade-invariant.

Assumption 3: Weight Differences Independence (WDI). We say that weight (relative) differences are independent of the fixed variable if:

$$\frac{w_{\alpha,j}}{w_{\beta,j}} = \frac{w_\alpha}{w_\beta} \quad \forall j \in [k],\ \forall (\alpha,\beta) \in \Omega^2$$

$$\frac{w_{\alpha,j}}{w_{\alpha,t}} = \frac{w_j}{w_t} \quad \forall \alpha \in \Omega,\ \forall (j,t) \in [k]^2$$

One way to model this assumption is to assume that the weight $w_{\alpha,j}$ is proportional to a standardized weight w_j as follows: $w_{\alpha,j} = \theta_\alpha w_j$ for every grade α and every rank j, where θ_α is a multiplicative coefficient capturing the "strength" of the grade α relatively to the hierarchy of grades as defined in the common language, and where w_j captures the marginal utility evolution structure which is thus common to all same-grade groups. This formulation actually implies the WDI hypothesis:

$$\frac{w_{\alpha,j}}{w_{\beta,j}} = \frac{\theta_\alpha w_j}{\theta_\beta w_j} = \frac{\theta_\alpha}{\theta_\beta}$$

$$\frac{w_{\alpha,j}}{w_{\alpha,t}} = \frac{\theta_\alpha w_j}{\theta_\alpha w_t} = \frac{w_j}{w_t}$$

Indeed, the weight difference related to grades only depends on the compared grades, and similarly for ranks. Obviously, we have $\theta_\alpha > \theta_\beta$ when α refers to a better grade than β. Precisely, by integrating the WDI hypothesis into the CGD condition ($w_{\alpha,j} > w_{\beta,t}\ \forall (j,t)$ when $\alpha \succ \beta$), we find that:

$$\theta_\alpha > \theta_\beta \frac{w_t}{w_j}\ \forall (j,t) \Rightarrow \theta_\alpha > \theta_\beta \frac{w_1}{w_k} > \theta_\beta$$

Thus, conditions on multiplicative coefficients become more restrictive. Moreover, the score of voter v_i could be transformed too:

$$f_i(W, M_i) = \sum_{\alpha \in \Omega} \sum_{j=1}^{|W \cap M_i(\alpha)|} \theta_\alpha w_j = \sum_{\alpha \in \Omega} \theta_\alpha \left(\sum_{j=1}^{|W \cap M_i(\alpha)|} w_j \right)$$

Finally, WDI implies that the individual score f_i is equal to a linear combination of classical weight sums similar to those in the approval voting analysis. Later, we show that the WDI assumption is useful when it comes to extend the Proportional Approval Voting, a particular Thiele rule, to the MJ setting.

3.2.3 The particular case of Proportional Approval Voting

In the very beginning of 2000s, Forrest Simmons featured one particular form of Thiele's general method that would become the most famous of them, the Proportional Approval Voting (PAV) [Kilgour, 2010]. This election rule states that the derived utility from the j^{th} representative is $w_j = 1/j$. The weights are harmonic numbers and are actually decreasing with the ranks. Thus, PAV captures the concept of decreasing marginal utility.

Now, we want to extend PAV to a MJ analysis. When supposing WDI and its previous modelling, we are able to isolate w_j and hence it is easy to apply PAV formulation: $w_{\alpha,j} = \theta_\alpha / j$. We suggest to name this extended rule Proportional Grade Voting (PGV).

Election rule 5.1: Proportional Grade Voting (PGV). Let F^{PGV} the GBC choice rule that maximizes the global score as defined by Proportional Grade Voting. Thus, for any profile M, F^{PGV} returns a winning committee such that:

$$F^{PGV}(M) = arg \max_{W \subset C} \left\{ \sum_{i=1}^{n} \left(\sum_{\alpha \in \Omega} \theta_\alpha \left(\sum_{j=1}^{|W \cap M_i(\alpha)|} \frac{1}{j} \right) \right) \right\}$$

But, why should we use harmonic numbers to model standardized weights? The authors [Brill and al, 2018] and [Lackner and Skowron, 2018] show that it was the only way (among Thiele's methods) to generate the proportional committee when facing party-list approval profiles. Let's precise that a party-list approval profile implies that every pair of approval subsets (A_i, A_j) are either equal $(A_i = A_j)$ or totally disjoint $(A_i \cap A_j = \emptyset)$, as if voters could vote for candidate lists, and only one of them (plurality voting). We talk about integral party-list profiles when each number of seats to be granted is a natural integer. Coming back to the MJ analysis, we can also find rare MJ profiles that are tantamount to such classical election and for which there is an obvious proportional solution.

First, we should reduce the MJ profile to an approval profile. Approval voting can be seen as a particular case of MJ voting where there are only two grades, "Approved" and "Disapproved". The common language should not necessarily be binary but voters should use only two grades, and the higher one would be considered as the approving grade. In order that approval and disapproval votes from each voter have the same value, the couple of grades selected by each of them should be exactly the same for every voter. If we denote α and β such grades, then every voter v_i has only two MJ subsets $M_i(\alpha)$ and $M_i(\beta)$. If $\alpha \succ \beta$, then $M_i(\alpha)$ can be considered as the approval subset of voter v_i.
Then, we should identify groups of candidates, that will be called "parties". It means that, in each group, every candidate receives the same grade for a given voter, as if they were only one candidate. For example, when $n = 100$, we can argue that the candidates c_1,c_2 and c_3 constitute a homogeneous group if they each receive a grade "Very Good" from the first fifty voters and if they each receive a grade "Acceptable" from the fifty last ones. We suppose that we identify R parties, and we denote $C_r \subset C$ the set of candidates belonging to the r^{th} party, such that $\bigcup_{r=1}^{R} C_r = C$ and $C_r \cap C_t = \emptyset$ for all $r \neq t$. If voter v_1 grants her higher grade to the second and the third parties only, we have $M_1(\alpha) = C_2 \cap C_3$ and $M_1(\beta) = C_1 \cap C_4 \cap \ldots \cap C_R$. This grouping into parties concretely means that if a voter α-grades a candidate of a party, she is obligated to assign the same grade to all its other members. Analytically, if $c_j \in M_i(\alpha)$ and $c_j \in C_r$, thus $C_r \subset M_i(\alpha)$.
Finally, each voter should vote for one and only one list. Thus, for every voter v_i, it exists $r(i) \in [R]$ such that $M_i(\alpha) = C_{r(i)}$ and, consequently, $M_i(\beta) = \bigcap_{1 \leq r \leq R}^{r \neq r(i)} C_r$ and $M_i(\gamma) = \emptyset$ for all $\gamma \in \tilde{\Omega} = \Omega \setminus \{\alpha, \beta\}$. For each couple of voters (v_i, v_j), we have either $r(i) = r(j)$ or $r(i) \neq r(j)$. In the first case, $M_i(\alpha) = C_{r(i)} = C_{r(j)} = M_j(\alpha)$, $M_i(\beta) = \bigcap_{1 \leq r \leq R}^{r \neq r(i)} C_r = \bigcap_{1 \leq r \leq R}^{r \neq r(j)} C_r = M_j(\beta)$ and $M_i(\gamma) = \emptyset = M_j(\gamma)$ for all $\gamma \in \tilde{\Omega}$. These results can be resumed under one equality: $M_i(\omega) = M_j(\omega)$ for all grades ω. In other words, voters of a same party have exactly the same α-subsets. In the second case, $M_i(\alpha) = C_{r(i)}$ and $M_j(\alpha) = C_{r(j)}$ with $C_{r(i)} \cap C_{r(j)} = \emptyset$ given that $r(i) \neq r(j)$, thus we obtain $M_i(\alpha) \cap M_j(\alpha) = \emptyset$. In addition, $M_i(\beta) \cap M_j(\beta) = (\bigcap_{1 \leq r \leq R}^{r \neq r(i)} C_r) \cap (\bigcap_{1 \leq r \leq R}^{r \neq r(j)} C_r) = C \setminus (M_i(\alpha) \cup M_j(\alpha))$, thus their intersection is composed of $(R-2)$ parties.

Proposition 4. *M is a party-list MJ profile if $\exists(\alpha, \beta) \in \Omega^2$ such that:*

- $M_i(\gamma) = \emptyset \quad \forall \gamma \in \Omega \setminus \{\alpha, \beta\}, \quad \forall i \in [n]$
- $\forall (i, j) \in [n]^2$, either $M_i(\alpha) = M_j(\alpha)$ or $M_i(\alpha) \cap M_j(\alpha) = \emptyset$
  - If $M_i(\alpha) = M_j(\alpha)$, then $M_i(\beta) = M_j(\beta)$
  - If $M_i(\alpha) \cap M_j(\alpha) = \emptyset$, then $M_i(\beta) \cap M_j(\beta) = C \setminus (M_i(\alpha) \cup M_j(\alpha))$

The Proportional Grade Voting could be considered as a "party-list proportional" election rule if it returns the corresponding proportional committee when facing party-list MJ profiles. Such a proportional committee is denoted $W^P(M)$ for every party-list MJ profile M. However, since political parties are "naturally" generated by voters' expressed preferences, they do not necessarily fit the number of seats they should be given, contrary to official party-list elections where each party P_r stand as many candidates as seats to be filled. Analytically speaking, $|P_r| = k$ and thus $|P_r| < s_r$ is impossible, whereas $|C_r| \in [m]$ and thus $|C_r| < s_r$ is possible. When the number of candidates in a party is inferior to the number of seats it deserves, the proportional committee cannot be generated. This issue is likely to occur when small parties receive a great number of votes and are consequently granted a high number of seats. For instance, suppose that $n = 100$, $m = 20$, $k = 10$ and that we can identify two parties C_1 and C_2 for having gathered 70% and 30% of votes respectively: candidates in C_1 all receive the higher grade α from 70 voters in V_1 and the lower grade β from 30 voters in V_2, and symmetrically, candidates in C_2 all receive the higher grade α from 30 voters in V_2 and the lower grade β from 70 voters in V_1. Proportionality states that seven seats should be given to candidates in C_1 and three seats should be granted to candidates in C_2. Nevertheless, assume that $|C_1| = 5$ and $|C_2| = 15$. Within such a configuration, we cannot give seven seats to the first party, as it is composed of five candidates only, and given that we cannot give more than one seat to any candidate. It is the same problem than in Subsection 2.3 where we were to give more than one seat to one candidate in order to respect strict proportionality. Thus, two solutions immediately appear. First, we can apply the LUD, in order to select the least unproportional committee. In our example, it would lead to give five seats to the five candidates in C_1 and give the five remaining seats to random candidates belonging to C_2. Second, we can implement the INP. The idea is to allocate seats to parties proportionally to votes while satisfying all candidate constraints $|C_r| < s_r$, that is to say, by normalizing to $|C_r|$ the number of seats granted to the party having the lowest quotient $(\frac{|C_r|}{s_r})$ and by applying the same normalization process to all other parties. Then, potential remaining seats are allocated among remaining parties – parties that still have unelected candidates– after having removed empty parties from the initial profile, and we should reiterate the same process until all seats are filled (Appendix C). In our example, it would lead to give 5 seats to the first party and $3(\frac{5}{7}) \approx 2$ seats to the second party, and the 3 remaining seats would be granted to the second party for gathering 100% of votes among remaining parties.

Axiom 11: Party-list proportionality (Plp). A GBC choice rule F is said party-list proportional if for any party-list MJ profile M,

$$F(M) = W^P(M)$$

This axiom is a minimal condition for an election rule to be considered proportional, but it is not sufficient. However, we can extend the scope of this axiom by defining a way to reduce some specific MJ profiles into a party-list one.Precisely, we can implement a grade bipolarization process, denoted Ψ, that reassigns each actual grade to another within a choice of two defined grades. In other words, this process is a mapping $\Psi : \Omega \to \{\alpha, \beta\}$. This process should be consistent with the hierarchy of grades as defined by the common language. For instance, when $\Omega = \{$"Very Bad";"Bad";"Mediocre";"Good";"Very Good"$\}$ and $\{\alpha, \beta\} = \{$"Good";"Mediocre"$\}$, if the grade "Bad" is reassigned to "Mediocre", then the grade "Very Bad" must necessarily be reassigned to "Mediocre", given that "Bad" $\succ$ "Very Bad" and "Good" $\succ$ "Mediocre". Analytically, when we have $\alpha \succ \beta$, a process Ψ is said **order-consistent** if for all $\omega_g \in \Omega$ such that $\Psi(\omega_g) = \beta$, we have $\Psi(\omega_h) = \beta$ whenever $\omega_g \succ \omega_h$, and if for all $\omega_g \in \Omega$ such that $\Psi(\omega_g) = \alpha$, we have $\Psi(\omega_h) = \alpha$ whenever $\omega_h \succ \omega_g$. Furthermore, the process must be useful, that is to say, it should not reassign all grades to the same pole. Otherwise, binarity would not be reached. Thus, a process Ψ is said **effective** if $\exists(\omega_g, \omega_h) \in \Omega^2$ such that $\Psi(\omega_g) \neq \Psi(\omega_h)$.

However, the implementation of a grade bipolarization process only satisfies the first condition of a party-list MJ profile, which is the binarity condition, but does not establish the two other conditions, which are identifying homogeneous groups of candidates and plurality voting. For instance, let consider the following MJ profile, denoted $\hat{M}$:

c_1	α	γ	β
c_2	β	α	γ
c_3	γ	β	α
	v_1	v_2	v_3

Where $\Omega = \{\alpha; \beta; \gamma\}$ and $\alpha \succ \beta \succ \gamma$. By order-consistency, we should have: $\Psi(\alpha) \geq \Psi(\beta) \geq \Psi(\gamma)$. Moreover, by effectiveness, the previous inequality implies that $\Psi(\alpha) = \alpha$ and $\Psi(\gamma) = \gamma$. Finally, the only free choice regards β's reassignment, for which we have two possibilities. Consequently, we can conceive only two different grade bipolarization processes, respectively denoted Ψ_1 and Ψ_2, such that $\Psi_1(\beta) = \alpha$ and $\Psi_1(\beta) = \gamma$. Precisely, Ψ_1 considers α and β as approving grades and γ as a disapproving one, whereas Ψ_2 considers α as the sole approving grade and β and γ as disapproving ones. In our example, whatever the grade bipolarization process we choose, it is impossible to gather candidates into parties. Indeed, gathering candidates requires that their respective "rows of grades" be identical. If we were to gather c_1 and c_2, we would have to verify $\Psi(\alpha) = \Psi(\beta)$ from v_1, $\Psi(\gamma) = \Psi(\alpha)$ from v_2, $\Psi(\beta) = \Psi(\gamma)$ from v_3, and thus $\Psi(\alpha) = \Psi(\beta) = \Psi(\gamma)$, which is impossible when considering effectiveness. We would have to verify the same inconceivable equality when aiming at gathering c_2 and c_3, or c_1 and c_3, or obviously, c_1 and c_2 and c_3. Thus, for the MJ profile we consider, there is no grade bipolarization process Ψ such that we can gather at least two candidates in one party. Fortunately, a party in the

sense of [Brill and al, 2018] and [Lackner and Skowron, 2018] can be composed of a sole candidate. It is a reasonable convention to the extent that groups of candidates are not really political parties but rather ideological groupings and it is totally conceivable that some candidates be alone if they defend eccentric ideas.

As a result, the grouping condition is always verified for any MJ profile. When each candidate has a unique row of grade and when there is no order-consistent and effective grade bipolarization process permitting to equalize the rows of grades of at least two candidates, then there are as many parties as candidates ($R = m$).

Theorem 2. Let g_i the vector of grades given by any voter $v_i \in V$. For every MJ profile M such that $g_i \neq g_j$ for every $(i, j) \in [n]^2$ and such that $\Psi(g_i) \neq \Psi(g_j)$ for every $(i, j) \in [n]^2$ and for every order-consistent and effective $\Psi : \Omega \to \{\alpha, \beta\}$, there is only $m!$ ways to gather candidates into parties and each identified party is a singleton:

$$C_{\eta(j)} = \{c_j\}$$

Where $\eta : [m] \to [m]$ is a bijection between candidates and parties.

Given that the binarity condition is systematically implemented by Ψ and that the grouping condition is always verified, the only problematic condition is the plurality voting condition, where each voter may vote for at most one party. In other words, this third condition states that each voter v_i may assign the approving grades $\Psi^{-1}(\alpha) = \{\omega \in \Omega, \Psi(\omega) = \alpha\}$ to at most one identified group of candidates, denoted $C_{r(i)}$. Considering again $\hat{M}$, where we can identify three parties C_1, C_2 and C_3 such that $C_j = \{c_j\}$ for every $j \in [3]$, the grade bipolarization process Ψ_1 does not permit to make each voter vote for at most one party:

c_1	α	γ	α
c_2	α	α	γ
c_3	γ	α	α
	v_1	v_2	v_3

With such a reassignment of grades, each voter approves two parties and disapproves one. Thus, we have not succeeded in reducing the profile to a plurality profile, but only to an approval profile. Conversely, the grade bipolarization process Ψ_2 permits to reach a plurality profile:

c_1	α	γ	γ
c_2	γ	α	γ
c_3	γ	γ	α
	v_1	v_2	v_3

In this reassigned profile, that can be denoted $\Psi_2(M)$, each voter votes for exactly one party: v_i votes for c_i and thus $r(i) = i$ for every $i \in [3]$. Therefore, there is at least one grade bipolarization process Ψ, especially Ψ_2, such that M be reduced to a party-list MJ profile.

Nevertheless, this is not true for all MJ profiles. There exist MJ profiles for which there is no grade bipolarization process permitting to reduce them to a party-list one. For instance, let consider the MJ profile $\tilde{M}$ such that:

c_1	α	α	α
c_2	α	β	γ
c_3	β	γ	β
	v_1	v_2	v_3

Once again, order-consistency and effectiveness imply that we can only consider two grade bipolarization processes Ψ_1 and Ψ_2, differing in the reassignment choice of β, and that we can simply identify three parties, each of them composed of one candidate. However, such a profile cannot be reduced to a party-list one. No matter which Ψ we choose, we remark that voter v_1 will always vote for more than one party. Precisely, she will always vote for C_1 and C_2, and she will also vote for C_3 when $\Psi(\beta) = \alpha$. Thus, it seems relevant to characterize the class of MJ profiles for which it exists at least one grade bipolarization process permitting to gather candidates into parties and making each voter approve at most one of these parties, as if they were in a plurality party-list election. Such MJ profiles could be called reducible, as they can be reduced to a party-list MJ profile.

Proposition 5. *M is a reducible party-list MJ profile if $\exists\ \Psi : \Omega \to \{\alpha, \beta\}$ satisfying order-consistency and effectiveness such that:*

- $C = \bigcup_{r=1}^{R} C_r$ where $\Psi(g_i(c_j)) = \Psi(g_i(c_{j'}))\quad \forall i \in [n],\quad \forall (c_j, c_{j'}) \in C_r^2,\quad \forall r \in [R]$
- $\forall\ v_i \in V$, $\exists\ r(i) \in [R]$ such that:
  - $\Psi(g_i(c_j)) = \alpha \quad \forall c_j \in C_{r(i)}$
  - $\Psi(g_i(c_j)) = \beta \quad \forall c_j \notin C_{r(i)}$

For every reducible party-list MJ profile M, it is thus possible to find a grade polarization process Ψ such that $\Psi(M)$ is a party-list MJ profile as defined in Proposition 4, and hence for which we can define a proportional committee, denoted $W^P(\Psi(M))$. We can then extend the scope of Axiom 11 to the class of reducible party-list MJ profiles.

Axiom 12: Extended Party-list proportionality (EPlp). Let Ψ an effective and order-consistent grade bipolarization process. A GBC choice rule F verifies extended party-list proportionality if for any reducible party-list MJ profile M:

$$F(M) = W^P(\Psi(M))$$

Conclusion

Through our article, we designed several GBC rules respecting a specific vision of proportionality, either political or welfarist. The diversity of what proportionality should mean is expressed through the axioms we design. For proportionality in terms of political influence, the equal power principle (Epp) is a desirable axiom. Symmetrically, when considering proportionality in terms of welfare, it is the equal utility principle (Eup) that should be verified. Moreover, some other proportionality axioms could be considered desirable whatever the way we define proportionality, since they are minimalist axioms; party-list proportionality (Plp) and extended party-list proportionality (EPlp) axioms. The rules we invent were either GBC choice rules such as Voting Gauge Methods (VGM) and the Iterated Normalization Process (INP), or GBC ranking rules such as the Least Unproportional Decision (LUD), the Rawlsian Committee (RC) rule or the Thiele's Grade Voting (TGV) rule, including Proportional Grade Voting (PGV). A considerable lack in our analysis is verifying which axioms are satisfied by each of these election rules. We expect welfarist rules to respect the Eup axiom and political rules to satisfy the Epp axiom, but all rules we create should verify the minimalist proportionality axioms and the desirable axioms designed in the first part. Moreover, if a GBC rule were satisfying both Eup and Epp axioms, we would argue that it is a proportional rule whatever the meaning given to proportionality. Thus, we call further research to conduct an axiomatic check for each GBC rule we created. Furthermore, it would also be interesting to determine the existing links between these axioms, that is to say, knowing if some axioms are compatible or not, if some axioms are implied by others or not etc.

However, proportionality is not the only desirable property when it comes to elect a fixed-size committee. According to [Elkind and al, 2017], multi-winner rules may follow three distinct desirable goals: proportionality, diversity and individual excellence. When considering diversity, we want each voter to have at least one representative, even if their opinion is extremely marginal. When considering individual excellence, we give priority to the most supported candidates, for being considered the most skillful. Unfortunately, these principles are rarely compatible. For instance, considering plurality voting with 3 seats to be allocated among 3 parties $\{P_1, P_2, P_3\}$ respectively gathering 66%, 33% and 1% of votes, proportionality would say to give two seats to P_1 and one seat to P_2, diversity would say to give one seat to each party, and individual excellence would say to give the three seats to P_1. It would be interesting that further research seek to extend MJ voting to the multi-winner setting and by targeting diversity or individual excellence principles, and not by considering proportionality as we did. For the diversity principle, we suggest further research extend the [Chamberlin and Courant, 1983] rule and, more generally, the Constant Threshold methods developed by [Fishburn and Pekec, 2004]. For the individual excellence principle, it seems that it would simply lead to giving the k seats to the k best graded candidates, or equivalently, maximizing the average grade of the least well-graded committee member, a generalization of the Maximin support method proposed by [Sánchez-Fernández and al, 2016].

Appendices

A. Discrete marginal political power

$$
\begin{aligned}
p_i(x_i + y, N + y) - p_i(x_i, N) &= (x_i + y)(\frac{k}{N + y}) - x_i \frac{k}{N} \\
&= \frac{N(x_i + y)k}{N(N + y)} - \frac{x_i k(N + y)}{N(N + y)} \\
&= \frac{k}{N(N + y)}[N(x_i + y) - x_i(N + y)] \\
&= \frac{k}{N(N + y)}[Nx_i + Ny - Nx_i - x_i y] \\
&= \frac{k}{N(N + y)}[Ny - x_i y] \\
&= \frac{ky(N - x_i)}{N(N + y)}
\end{aligned}
$$

B. Programming the LUD

The following R Script should be applied through the R software.

```
#1. Modeling the Approval Profile

A <- data.frame(row.names=c("c1","c2","c3","c4","c5","c6"))

A[,1] <- c(1,1,0,1,0,0)
A[,2] <- c(0,1,1,0,0,0)
A[,3] <- c(1,0,1,1,1,0)
A[,4] <- c(0,1,1,0,0,1)
A[,5] <- c(0,0,0,1,0,0)
A[,6] <- c(1,1,0,0,0,1)
A[,7] <- c(0,0,1,1,0,0)
A[,8] <- c(1,1,0,1,0,1)

#2. Determining all possible committees

  #2.1 When the committee size is not determined (64 possibilities)
S <- data.frame(row.names=seq(1,64,1))

for (j in 1:6) {
  J <- 64/(2^(j-1))
  L <- rep(0,J)

  for (t in 1:J) {
    if (t<(J/2)+1) L[t] <- 1
  }

  S <- cbind(S, rep(L,2^(j-1)))
}

  #2.2 Identifying the size of each committee
k <- rep(0,64)

for (t in 1:64) {
  k[t] <- sum(S[t,])
}

  #2.3 Fixing the committee size to 4 (15 possibilities)
Z <- S[k==4,]
rownames(Z) <- seq(1,nrow(Z),1)

#3. Computing the political power of each voter

  #3.1 Number of approved candidates by voter
x <- rep(0,8)

for (i in 1:8) {
  x[i] <- sum(A[,i])
```

```
}

 #3.2 Number of voting points gathered by each candidate
Q <- rep(0,6)

for (j in 1:6) {
  Q[j] <- sum(A[j,]/x)
}

 #3.3 Matrix of political power for each voter in each possible committee
p <- matrix(nrow = 15, ncol = 8)

for (y in 1:15) {
  for (i in 1:8) {
    p[y,i] <- sum(((A[,i]/x[i])/Q)*Z[y,])
  }
}

#4. Determining the winning committee

  #4.1 Defining the gaps matrix
e <- p - 4/8

e2 <- e*e

  #4.2 Measuring the collective proportionality loss in each committee
Cpl <- rep(0,15)

for (y in 1:15) {
  Cpl[y] <- sum(e2[y,])
}

  #4.3 Minimizing the collective proportionality loss
W <- Z[Cpl==min(Cpl),]
W
```

C. Generalizing the INP

Initial allocation
Consider a party-list MJ profile M where R parties C_r can be identified. Let $n_r = |\{v_i \in V,\ M_i(\alpha) = C_r\}|$ the number of voters who assign the approving grade to the r^{th} party and $s_r = (\frac{n_r}{n})k$ the number of seats that should theoretically be granted to C_r. Assume that parties in $B \subset [R]$ cannot fill all the seats they theoretically deserve: $|C_b| < s_b\ \forall b \in B$. Among these parties, let denote d the party for which the seats quotient is minimal:

$$d = arg\min_{b \in B} \left\{ \frac{|C_b|}{s_b} \right\}$$

Thus, if we denote $\tilde{s}_r$ the actual seats given to any party r, then $\tilde{s}_d = |C_d|$ seats should be given to the party C_d and $\tilde{s}_r = s_r(\frac{|C_d|}{s_d})$ should be granted to any other party C_r. Unelected candidates of party C_r are gathered in Γ_r. Obviously, $\Gamma_d = \emptyset$. The number of remaining seats after this first allocation is denoted k_2 and is equal to:

$$k_2 = k - \sum_{r=1}^{R} s_r \left(\frac{|C_d|}{s_d} \right) = k - \sum_{r=1}^{R} |C_d| \left(\frac{n_r}{n_d} \right)$$

These remaining seats have to be proportionally allocated among remaining parties, that is to say, parties for which $|\Gamma_r| > 0$. We know that C_d is the only party having all its candidates already elected, but there could be ties when determining the lowest seats quotient. Therefore, we suggest to denote $D \subset B$ the set of parties that minimizes such a quotient. Each of these parties could be taken for normalization, but whatever the party we choose, all these parties will bind their candidate constraint and hence we will have $|\Gamma_d| = 0$ for every party $d \in D$. Finally, after the first allocation, there still are k_2 seats to be allocated among $(R - |D|)$ parties.

Following allocations
As there may be several allocations, let denote $s_{r,z}$ the number of seats that should theoretically be granted to the party C_r within the z^{th} allocation, $\Gamma_{r,z}$ the set of unelected candidates from that party before the z^{th} allocation, with $\Gamma_{r,1} = C_r$, B_z the set of parties for which $|\Gamma_{b,z}| < s_{b,z}$, and $D_z \subset B_z$ the set of parties that minimizes $(\frac{|\Gamma_{b,z}|}{s_{b,z}})$. Furthermore, let k_z the number of remaining seats and R_z the number of remaining parties before the z^{th} allocation. Precisely, $R_z = R_{z-1} \setminus D_{z-1}$ and $R_1 = [R]$.

After the first allocation, if there still are seats to be allocated, we should remove the empty parties D_1 from the initial profile to obtain the new proportionality structure between remaining parties R_2. According to the new profile, each remaining party $r \in R_2$ should be given $s_{r,2}$ such that:

$$s_{r,2} = \left(\frac{n_r}{n - \sum_{d_1 \in D_1} n_{d_1}} \right) k_2 = \left(\frac{n_r}{n - \sum_{d_1 \in D_1} n_{d_1}} \right) \left(k - \sum_{r \in R_1} |C_{d_1}| \left(\frac{n_r}{n_{d_1}} \right) \right)$$

However, it is possible that some parties do not have enough remaining candidates to fill all the seats they deserve: $|\Gamma_{b,2}| < s_{b,2}\ \forall b \in B_2$. In that case, we should give $\tilde{s}_{d_2,2} = |\Gamma_{d_2,2}|$ seats to any party $d_2 \in D_2$, the set of parties in B_2 that have the lowest seats quotient, and $\tilde{s}_{r,2} = s_{r,2}(\frac{|\Gamma_{d_2,2}|}{s_{d_2,2}})$ seats to any other party $r \in R_3$. Finally, for every $r \in R_2$:

$$\tilde{s}_{r,2} = s_{r,2}\left(\frac{|\Gamma_{d_2,2}|}{s_{d_2,2}}\right) = |\Gamma_{d_2,2}|\left(\frac{n_r}{n_{d_2}}\right)$$

Finally, the number of remaining seats after the second allocation is:

$$k_3 = k_2 - \sum_{r\in R_2} |\Gamma_{d_2,2}|\left(\frac{n_r}{n_{d_2}}\right)$$

$$\Leftrightarrow k_3 = k - \sum_{r\in R_1} |C_{d_1}|\left(\frac{n_r}{n_{d_1}}\right) - \sum_{r\in R_2} |\Gamma_{d_2,2}|\left(\frac{n_r}{n_{d_2}}\right)$$

$$\Leftrightarrow k_3 = k - \sum_{j=1}^{2}\sum_{r\in R_j} |\Gamma_{d_j,j}|\left(\frac{n_r}{n_{d_j}}\right)$$

If $k_3 > 0$, then it should have a third allocation among the parties in R_3. Theoretical seats are:

$$s_{r,3} = \left(\frac{n_r}{n - \sum_{j=1}^{2}\sum_{d_j\in D_j} n_{d_j}}\right)k_3 = \left(\frac{n_r}{n - \sum_{j=1}^{2}\sum_{d_j\in D_j} n_{d_j}}\right)\left(k - \sum_{j=1}^{2}\sum_{r\in R_j} |\Gamma_{d_j,j}|\left(\frac{n_r}{n_{d_j}}\right)\right)$$

And if $|B_3| > 0$, seats that are really granted are:

$$\tilde{s}_{r,3} = s_{r,3}\left(\frac{|\Gamma_{d_3,3}|}{s_{d_3,3}}\right) = |\Gamma_{d_3,3}|\left(\frac{n_r}{n_{d_3}}\right)$$

General allocation

Let consider the z^{th} allocation. We have k_z remaining seats to allocate among $|R_z|$ remaining parties. Theoretical seats are:

$$s_{r,z} = \left(\frac{n_r}{n - \sum_{j=1}^{z-1}\sum_{d_j\in D_j} n_{d_j}}\right)\left(k - \sum_{j=1}^{z-1}\sum_{r\in R_j} |\Gamma_{d_j,j}|\left(\frac{n_r}{n_{d_j}}\right)\right)$$

Supposing $|B_z| > 0$, seats that are really granted are:

$$\tilde{s}_{r,z} = |\Gamma_{d_z,z}|\left(\frac{n_r}{n_{d_z}}\right)$$

Now, suppose there are Z allocations, that is to say, the Z^{th} allocation is the latest. It implies that $|B_Z| = 0$, otherwise theoretical seats would have not been given to each remaining party in R_Z and a $(Z+1)$ allocation would have been necessary to allocate the k_{Z+1} remaining seats. Thus, we can divide the R parties into Z mutually exclusive groups:

$$[R] = \left(\bigcup_{j=1}^{Z-1} D_j\right) \cup R_Z$$

All parties in D_j face the j first allocations and are empty after the j^{th} allocation. Thus, they were given seats j times. By denoting S_r the total number of seats really given to any party $r \in [R]$ through all possible allocations, we can state that, for every party $d_j \in D_j$ and for every $j \in [Z-1]$:

$$S_{d_j} = \sum_{z=1}^{j} \tilde{s}_{d_j,z} = \sum_{z=1}^{j} |\Gamma_{d_z,z}| \left(\frac{n_{d_j}}{n_{d_z}} \right)$$

For $j > 1$, we can write:

$$S_{d_j} = \left(\sum_{z=1}^{j-1} |\Gamma_{d_z,z}| \left(\frac{n_{d_j}}{n_{d_z}} \right) \right) + |\Gamma_{d_j,j}|$$

For $j = 1$, we can obtain a simpler formulation:

$$S_{d_1} = |\Gamma_{d_1,1}| = |C_{d_1}|$$

All parties in R_Z are given seats in each possible allocation. Precisely, they are given normalized seats in the $(Z-1)$ first allocations and theoretical seats in the last allocation. Thus, we can state that, for every party $r \in R_Z$:

$$S_r = \left(\sum_{z=1}^{Z-1} \tilde{s}_{r,z} \right) + s_{r,Z}$$

$$\Leftrightarrow S_r = \left(\sum_{z=1}^{Z-1} |\Gamma_{d_z,z}| \left(\frac{n_r}{n_{d_z}} \right) \right) + \left(\frac{n_r}{n - \sum_{j=1}^{Z-1} \sum_{d_j \in D_j} n_{d_j}} \right) \left(k - \sum_{j=1}^{Z-1} \sum_{r \in R_j} |\Gamma_{d_j,j}| \left(\frac{n_r}{n_{d_j}} \right) \right)$$

Finally, a proportional committee obtained by INP, denoted W^{INP}, is a committee composed of R smaller subcommittees W_r, one for each party $r \in [R]$, where $|W_r| = S_r$ and where there are $\binom{|C_r|}{S_r}$ possibilities for W_r. Precisely, W^{INP} can be decomposed as follows:

$$W^{INP} = \left(\bigcup_{j=1}^{Z-1} \left(\bigcup_{d_j \in D_j} W_{d_j} \right) \right) \cup \left(\bigcup_{r \in R_Z} W_r \right)$$

We denote $\mathbb{W}^{INP}$ the set of all proportional committees that can be obtained by implementing the INP. Since there is only one possible combination for all W_{d_1}, we deduce that:

$$\left| \mathbb{W}^{INP} \right| = \left(\prod_{j=2}^{Z-1} \left(\prod_{d_j \in D_j} \binom{|C_{d_j}|}{S_{d_j}} \right) \right) \left(\prod_{r \in R_Z} \binom{|C_r|}{S_r} \right)$$

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